Critical curves in conformally invariant statistical systems

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Abstract

We consider critical curves — conformally invariant curves that appear at critical points of two-dimensional statistical mechanical systems. We show how to describe these curves in terms of the Coulomb gas formalism of conformal field theory (CFT). We also provide links between this description and the stochastic (Schramm-) Loewner evolution (SLE). The connection appears in the long-time limit of stochastic evolution of various SLE observables related to CFT primary fields. We show how the multifractal spectrum of harmonic measure and other fractal characteristics of critical curves can be obtained.

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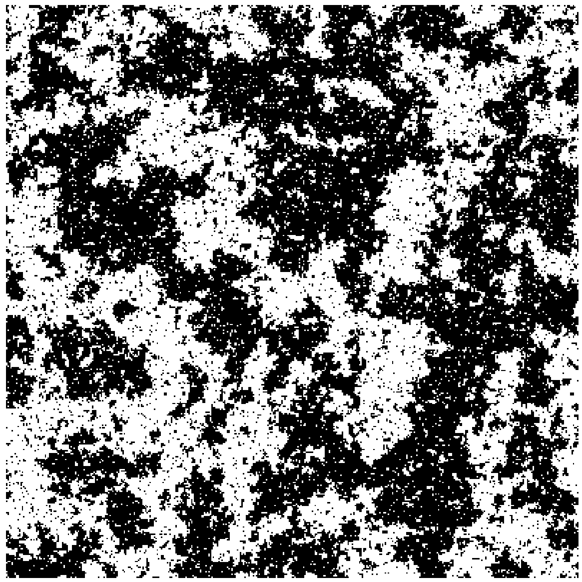

Figure 1: A critical Ising model. Black and white represent ± 1 spins. The cluster boundaries furnish an example of critical curves (after http://www.ibiblio.org/e-notes/Perc/contents.htm).

1 Introduction

The continuous limit of two-dimensional (2D) critical statistical systems exhibits conformal invariance which proved to be a very useful and computationally powerful concept [1, 2, 3]. Because of the conformal invariance a system with a boundary can be studied in any of the topologically equivalent domains. We will consider critical systems in domains with the topology of an annulus which for the ease of computations can be mapped by a conformal transformation onto the upper half plane with a puncture.

Critical 2D systems can be described in terms of fluctuating curves (loops, if they are closed) which can be understood as domain walls or boundaries of clusters (e.g. Fig. 1). The exact definition depends on the system in question and typical examples include the boundaries of the Fortuin-Kasteleyn clusters in the q-state Potts model, the loops in the $O(n)$ model, and the cluster boundaries in a critical percolation system. We use the term critical curves to describe any of these objects.

Critical curves are drawn from a statistical thermal ensemble and because of the scale invariance at criticality they are self-similar random fractals.

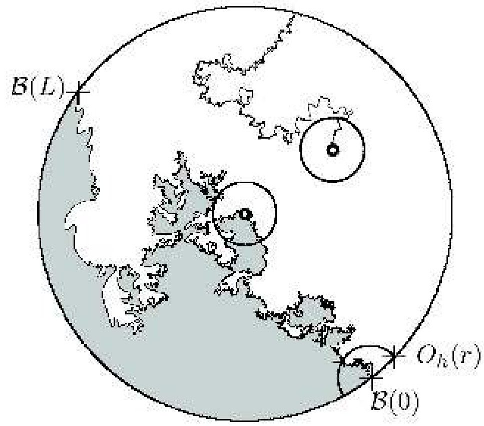

Figure 2: A critical curve is conditioned to start from 0, pass through a puncture and end at L on the boundary. Another critical curve is conditioned to end at a puncture in the bulk.

A large scope of their geometrical properties can be investigated. As an example, one can study the scaling of the electric field near a critical curve which is assumed to be the boundary of a conducting and charged cluster. Averaged over the fluctuations of the curve (we denote such averages by $\prec \ldots \succ$), the moments of the electric field E scale with the distance r to the curve as $\prec E^h(r) \succ \sim C(h) r^{\Delta(h)}$ with a non-trivial multifractal exponent $\Delta(h)$ first computed in [4, 5]. On the one hand, this exponent is related to the multifractal spectrum of the harmonic measure of the critical curve (and, in particular, determines its fractal dimension). On the other hand, $\Delta(h)$ happens to be the gravitationally dressed dimension h introduced in [7], thus forming an interesting link to quantum gravity. The determination of the h-dependent prefactor $C(h)$ is also interesting and, to our knowledge, has not been done yet. One can also study more complicated correlators of moments of the electric field measured at different points: $\prec E^{h_1}(z_1) E^{h_2}(z_2) \ldots \succ$ (we use complex coordinates z to denote points in the plane). Such objects probe local properties of critical curves.

In this paper, which extends our results published in Ref. [6], we consider both the CFT and SLE approaches to stochastic geometry of critical curves and elaborate on the relation between the two. We also obtain the results of [4, 5] for multifractal exponents using CFT without reference to quantum gravity.

2 Creation of critical curves and conformal invariance

As the statistical system fluctuates critical curves come and go in a random fashion and this makes them difficult to study. We can, however, choose a point in the bulk or on the boundary and remove from the partition function all realizations except for those in which there is a curve passing through this point. The physical meaning of such conditioning is transparent. Consider a critical Ising model in the upper half plane and define critical curves to be the boundaries of clusters of spins of the same sign. As a boundary condition, we demand that all spins on the negative real axis are $+1$, and all those on the positive real axis are -1. This condition guarantees that in all realizations there will be a critical curve growing from the origin.

This is just one of many imaginable ways of conditioning the system but it illustrates the point: the existence of a curve growing from a point on the boundary is ensured by a change of the boundary condition at this point. To condition the curve to pass through a bulk point one can insert there a puncture, thus effectively turning it into a boundary point (Fig. 2).

Critical curves are conformally invariant [1, 2] (and are sometimes also called conformally invariant curves) in the sense depicted in Fig. 3. Define a system in two different but topologically equivalent domains A and B and introduce a conformal transformation $f(z)$ which maps A onto B. We consider the ensemble of loops in both domains and also the ensemble C of images of loops produced by the conformal map $f(z)$. The lattice in C is warped by the conformal transformation which makes it different from the system in B. Nevertheless, it may happen that the loops in B and C are statistically identical to each other, that is, they are different random realizations drawn from the same statistical ensemble. In this case we say that the ensemble of loops possesses conformal invariance.

The continuous limit of critical systems with conformal invariance is described by conformal field theory (CFT) which is characterized by a central charge c. In this paper we consider only theories with $c \leqslant 1$. For $-2 \leqslant c \leqslant 1$ they can be obtained as a continuous limit of the lattice $O(n)$ model.

The central charge itself does not completely determine the critical curves: their behavior crucially depends on boundary conditions. A CFT with $c \leqslant 1$ can be represented as a theory of a Bose field with the Neumann or the Dirichlet boundary conditions. From the point of view of the $O(n)$ model, different boundary conditions correspond to dense and dilute phases in which the behavior of critical curves is different. An impressive progress in under-

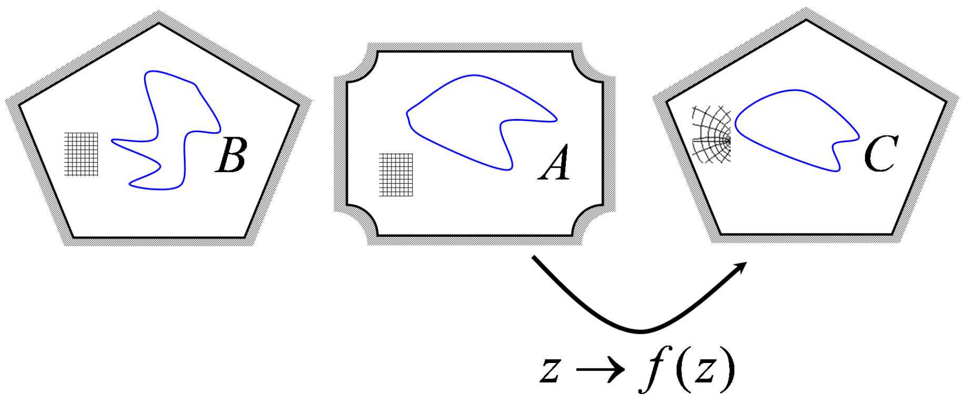

Figure 3: A and B are critical systems defined in different domains. The system C is the image of A under the conformal transformation $f(z)$ which maps domain A onto domain B. Conformal invariance is the statistical identity of B and C.

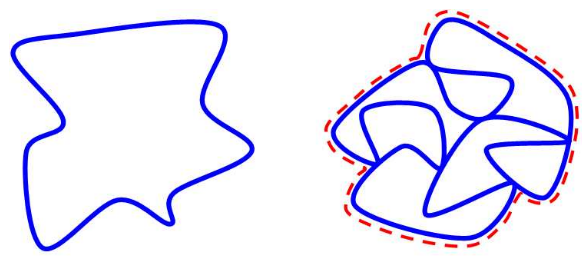

Figure 4: Schematic behavior of dilute (left) and dense (right) loops. The external perimeter of the dense curve is a dilute curve.

standing stochastic geometry of critical curves was achieved in the stochastic Loewner evolution (also known as Schramm-Loewner evolution, or SLE) [8]-[19]. It can be deduced that curves in the theory with the Dirichlet boundary condition (dilute phase) are almost surely simple. This means that the probability for them to touch themselves or each other vanishes. Those with the Neumann boundary condition for $c \geqslant -2$ (dense phase) almost surely touch themselves and each other (Fig. 4). For $c < -2$ the curves are plane-filling.

There exists a duality relation between dense and dilute phases [20]. In particular, the external perimeter of a dense curve is a dilute curve [5]. Thus, studying the dilute phase also answers some questions in the dense one. In CFT this duality is expressed by the fact that both phases can have

the same central charge and the difference lies in the boundary conditions.

3 Boundary operators and stochastic geometry

Consider a system in the dilute phase occupying the upper half plane and conditioned to contain a critical curve γ connecting the points 0 and L on the real axis (see Fig. 2). As was mentioned above, this will be the case if the boundary condition changes at these two points. Such a change is produced by the insertion of a special boundary condition changing operator which we can call a curve-creating operator. In statistical models with $c \leqslant 1$ such as the $O(n)$ model it is known [21] and we will discuss it later. The correlator of two such operators

$$\langle \mathcal{B}(0)\mathcal{B}(L)\rangle_{\mathbb{H}} = \frac{Z(0,L)}{Z}.$$

is determined by the partition function $Z(0,L)$ with boundary conditions changing at 0 and L. Here Z is the partition function of the system with uniform boundary conditions. The subscript of the correlator always refers to the domain of definition. We note in passing that by fusing together several such $\mathcal{B}$'s we obtain other boundary condition changing operators whose insertion produces multiple curves growing from a point on the boundary.

This correlation function can be interpreted in terms of stochastic geometry of the curve. The interpretation involves a two-step averaging (similar arguments can be identified in [10, 22]). In the first step we pick a particular realization of the curve γ. Then γ is the boundary separating two independent systems — the interior and the exterior of γ. In both these systems we can sum over the microscopic degrees of freedom to obtain the partition functions Z_γ^{int} and Z_γ^{ext}, respectively. These are stochastic objects that depend on the fluctuating geometry of γ.

In the second step we average over the ensemble of curves of γ. We thus obtain

$$Z(0,L) = \prec Z_\gamma^{\text{int}} Z_\gamma^{\text{ext}} \succ ,$$

where $\prec \ldots \succ$ stands for averaging over the shape of γ.

We further consider the insertion of an additional field $\mathcal{O}(z)$ on the boundary and a distant field $\Psi(\infty)$ such as to make non-zero the correlation function

$$\langle \mathcal{B}(0)\mathcal{O}(z)\mathcal{B}(L)\Psi(\infty)\rangle_{\mathbb{H}} \tag{1}$$

The exact form of $\Psi(\infty)$ is not important, for example we can choose it as $\mathcal{O}(\infty)$. If we are only interested in the z-dependence of this correlation function in the limit $|z| \ll |L|$, we can fuse together the distant fields: $\mathcal{B}(L)\Psi(\infty) :\to \Psi(\infty)$ and therefore consider a three-point function

$$\langle \mathcal{B}(0)\mathcal{O}(z)\Psi(\infty)\rangle_{\mathbb{H}},$$

In the two-step averaging procedure we rewrite (1) as an average over the fluctuating geometry of γ: $\prec\langle \mathcal{O}(z)\Psi(\infty)\rangle_\gamma Z_\gamma^{\rm int} Z_\gamma^{\rm ext}\succ$, where $\langle \mathcal{O}(z)\Psi(\infty)\rangle_\gamma$ stands for the correlation function in the exterior of γ. In the limit $|z| \ll |L|$ this correlation function is statistically independent from the other two factors, and we are left with the scaling relation:

$$\langle \mathcal{B}(0)\mathcal{O}(z)\mathcal{B}(L)\Psi(\infty)\rangle_{\mathbb{H}} \sim \prec\langle \mathcal{O}(z)\Psi(\infty)\rangle_\gamma\succ \tag{2}$$

The two-step averaging can be extended to multiple operators. If we choose $\mathcal{O}(z)$ a primary field, this relation yields the statistics of harmonic measure of critical curves (see Sec. 6).

4 Loop models and the Bose field

4.1 Dense phase

The relation between critical curves and operators of a CFT with a boundary is most transparent in the representation by a Bose field $\varphi(z, \bar{z})$ [3, 20, 23, 24]. This representation is commonly known as the Coulomb gas method. As in the case of any field theory, its derivation from the lattice model is not rigorous but is *a posteriori* justified by the results.

A large variety of 2D lattice statistical models (percolation, q-state Potts, $O(n)$, XY, the solid-on-solid and other similar models) can be defined in terms of a system of random curves, usually closed loops unless special boundary conditions are imposed. Each loop has a fixed statistical weight in the ensemble. Mappings between specific lattice models and loop models are described in detail in many reviews [20, 25, 26, 27]. Here we consider the $O(n)$ model as a representative example. It covers a large range of central charges ($-2 \leqslant c \leqslant 1$) and its dilute and dense phases correspond to the Dirichlet and Neumann boundary conditions of CFT. The model can be defined on the honeycomb lattice on a cylinder (such as a carbon nanotube) in terms of closed loops (Fig. 5):

$$Z_{O(n)}^{\rm cyl} = \sum_{\rm loops} x^L n^N. \tag{3}$$

Here x is a parameter related to the temperature, L is the total length of all loops and N is the number of loops. The parameter n is not necessarily an integer.

The $O(n)$ model has a critical point at

$$x = x_c(n) = \frac{1}{\sqrt{2 + \sqrt{2 - n}}}$$

for all n in the range $-2 \leqslant n \leqslant 2$. At the critical point the typical length of a loop diverges but the loops are dilute in the sense that the fraction of the vertices visited by them is zero. For $x > x_c(n)$ the typical length of the loops still diverges so that they are still critical but visit a finite fraction of the sites. This is the dense phase of the model. In the continuous limit dilute loops are simple and dense ones can touch themselves and each other. Finally, at zero temperature ($x = \infty$) the loops go through every point on the lattice. This is the fully packed phase. In the continuous limit the loops become plane filling.

The continuous description of the critical behavior of the $O(n)$ model by a local field theory is best understood in the dense phase. We need to have a description of the system in terms of local weights on the lattice and this is done as follows. One randomly orients the loops and assigns a weight to each orientation. In order to reproduce the partition function (3) the sum of the weights for two orientations of every loop should be n. This is achieved by giving a local complex weight $e^{\pm i e_0 \pi/6}$ to each lattice site where an oriented loop makes a right (left) turn. The weight of an oriented closed loop is the product of weights on all sites that it visits. For a closed loop which does not wrap around the cylinder this product equals $e^{\pm i e_0 \pi}$ since the difference between the numbers of right and left turns is ± 6. The sum over the two orientations gives the real weight n for an un-oriented loop if we choose

$$n = 2 \cos \pi e_0.$$

The range of $-2 \leqslant n \leqslant 2$ is covered once by $0 \leqslant e_0 \leqslant 1$. To describe both the dilute and the dense phases, however, we will need to allow for a wider range $-1 \leqslant e_0 \leqslant 1$, with positive e_0 for the dense phase and negative e_0 for the dilute phase.

The situation is different for the loops which wrap around the cylinder. For them the difference between the numbers of right and left turns is 0 and they are wrongly counted with weight 2 instead of n. To correct this, one defines for each configuration of oriented loops a real height variable H

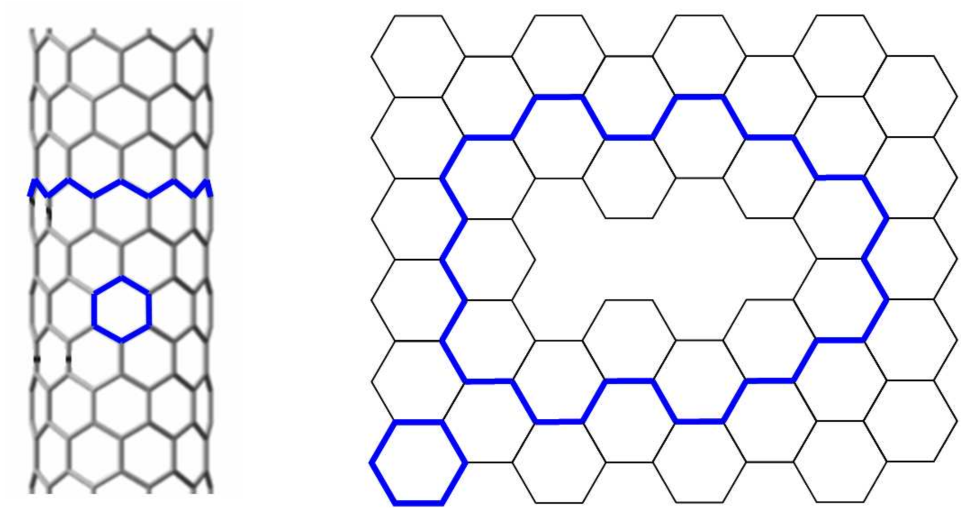

Figure 5: If a loop wraps around the cylinder, the difference between the numbers of left and right turns is 0, otherwise it is ± 6. On a flat domain of the same topology — annulus, — this difference is ± 6 for all loops.

which resides on the dual lattice and takes discrete values conventionally chosen to be multiples of π. We start at some reference point where we set $H = 0$, and then every time we cross an oriented loop, we change H by $\pm\pi$ depending on whether we cross the loop from its left to its right or vice versa. Then one introduces an additional factor $e^{ie_0(H(+\infty)-H(-\infty))}$ into each term of the partition function, where $\pm\infty$ stand for two chosen points on the two boundaries of the system (the bases of the cylinder). This factor is 1 if there are no wrapping loops, so it does not affect the weights of non-wrapping loops. But since each wrapping loop changes $H(+\infty) - H(-\infty)$ by $\pm\pi$, this factor turns into the proper weight n for each wrapping loop. Thus, we represent the partition function (3) in the form

$$Z^{\mathrm{cyl}}_{O(n)} = \sum_{\text{oriented loops}} x^L e^{ie_0(H(+\infty)-H(-\infty))} \prod_{\text{sites}} e^{\pm ie_0\pi}. \tag{4}$$

We note that the insertion of the fields at the boundaries is brought about not only by the topology of the domain: in a flat annulus, although its topology is the same as cylinder's, all loops are counted with the weight n as it is, so that no extra factors are needed (Fig. 5).

When we pass to the continuous limit, the lattice height function H is

coarse-grained and becomes a fluctuating compactified scalar field $h(z, \bar{z})$:

$$h \simeq h + 2\pi.$$

Upon coarse-graining the $O(n)$ loops on a cylinder become level lines of the field h. In the dense phase this field is believed to be Gaussian, that is, its action contains the term $(g/4\pi) \int d^2x\, (\nabla h)^2$, where the fluctuation strength parameter g is yet to be determined. The boundary terms of (4) have to be added in any geometry in the form of an oriented boundary integral $(ie_0/2\pi) \int_{\partial D} dl\, Kh$ where K is the geodesic curvature of the boundary. This is consistent with (4).

A similar situation occurs if the critical system lies on a surface with curvature which microscopically can be viewed as existence of defects on the honeycomb lattice (pentagons and heptagons correspond to positive and negative curvature, correspondingly). The correct weight for a loop that surrounds a region of non-zero Gaussian curvature R is obtained if we include in the action the so-called background charge term $(ie_0/8\pi) \int_D d^2x\, Rh$.

The determination of the coupling constant g can be done either by comparison with the exact solutions of a six-vertex model related to the $O(n)$ model or by an elegant argument of [28, 29]. Another necessary term in the action is the *locking potential* of the form $\lambda \int d^2x\, V(h)$ which favors the discrete values $h = 0, \pi$. It must be, therefore, a π-periodic function of h, the most general form of it being $V = \sum_{l \in \mathbb{Z}, l \neq 0} v_l e^{2ilh}$. We treat the locking potential perturbatively. In the zeroth order ($\lambda = 0$), each term of V is a vertex operator of dimension [20] $x_l = (2/g) l(l - e_0)$. An operator with $x_l < 2$ is relevant. For the perturbative approach to be applicable, the locking potential should contain no relevant terms. Furthermore, the most relevant term should be marginal. This fixes g. For $0 < e_0 < 1$ the most relevant term is $l = 1$ and this gives the relation

$$e_0 = 1 - g, \qquad\qquad n = -2\cos \pi g. \tag{5}$$

Note that $0 < g < 1$, as it is expected in the dense phase.

This description is not valid in the dilute phase. There H does not renormalize into a Gaussian field. A manifestation of this is that it is impossible to make the locking potential marginal for $1 \leqslant g \leqslant 2$. Naively, this requires taking $-1 < e_0 < 0$, but then one needs to pick $l = -1$ term as the most relevant and it still gives us $g = 1 + e_0 = 1 - |e_0| < 1$.

The point $g = 1$ separates the phases and is somewhat special: there the $l = 1$ and $l = -1$ terms in the locking potential have the same dimension and we have to keep them both. The relation between Gaussian fields and

critical curves has been made rigorous in [30] which in our language describes the case $n = 2$, $g = 1$.

We now introduce the parametrization

$$g = \frac{4}{\kappa}.$$

The range $0 < g \leqslant 2$ where the $O(n)$ model exhibits critical behavior corresponds to $2 \leqslant \kappa < \infty$. Smaller values $0 < \kappa < 2$ correspond to multi-critical points in the $O(n)$ model but also make sense for other critical systems. Notice that $\kappa < 4$ and $\kappa > 4$ describe the dilute and the dense phases correspondingly, while $\kappa = 4$ gives the point $g = 1$ separating the two phases. Readers familiar with SLE will recognize that this is similar to the SLE phases [31]. In fact, κ can be identified with the SLE parameter. For $\kappa \leqslant 4$ SLE curves are simple and for $4 < \kappa < 8$ they have double points.

In the literature it is customary to rescale the field $\varphi = \sqrt{2g}h$ and fix the coupling constant g to be $1/2$ at the expense of varying the compactification radius of φ:

$$\mathcal{R} = \sqrt{\frac{8}{\kappa}}. \tag{6}$$

This is the normalization that we adopt from now on. Up to the locking potential one then writes the partition function as

$$Z_{O(n)} \sim \int \mathcal{D}\varphi \cos\Big[\frac{\sqrt{2}\alpha_0}{2\pi} \oint_{\partial D} dl\, K\varphi\Big] e^{-\int (\nabla\varphi)^2 \frac{d^2x}{8\pi}}, \tag{7}$$

where the relation (5) reads[1]

$$2\alpha_0 = \frac{\sqrt{\kappa}}{2} - \frac{2}{\sqrt{\kappa}} > 0.$$

The marginal locking potential now reads $\lambda \int d^2x\, e^{i\sqrt{2}\alpha_+\varphi}$, where we used the notation

$$\alpha_+ = \frac{\sqrt{\kappa}}{2}, \quad \alpha_- = -\frac{2}{\sqrt{\kappa}}. \tag{8}$$

The partition function written in this way remains valid for any flat geometry with any number of boundaries.

[1]The notation differs from [6], where $\alpha_0 < 0$ in the dense phase. As a consequence, in the Kac notation $\psi_{r,s}$ of curve-creating operators $r \leftrightarrow s$.

Note that $\alpha_\pm$ satisfy simple relations $\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2+1}$. The locking potential is treated as a perturbation: in each order of the perturbation theory a number of screening charges $\int d^2x\, e^{i\sqrt{2}\alpha_+\varphi}$ is inserted into all correlation functions so as to satisfy the neutrality condition [32]. Note that the second screening charge $\int d^2x\, e^{i\sqrt{2}\alpha_-\varphi}$, although also of zero dimension, is not found by the argument on the lattice — it is incompatible with the compactification radius.

If the domain's Gaussian curvature R is not zero (as it happens when heptagons or pentagons are inserted as defects of the lattice) an additional term $i\frac{\sqrt{2}\alpha_0}{4\pi}\int d^2x\, R\varphi$ is needed.

Due to the symmetry $\varphi \to -\varphi$ we can rewrite the partition function as a local field theory $Z_{O(n)} \sim \int \mathcal{D}\varphi e^{-S}$, but with a *complex* action $S = \mathcal{A} + \lambda \int_D d^2x\, e^{i\sqrt{2}\alpha_+\varphi}$ where

$$\mathcal{A} = \frac{1}{8\pi}\int_D d^2x\,(\nabla\varphi)^2 + i\frac{\sqrt{2}\alpha_0}{2\pi}\int_{\partial D} dl\, K\varphi, \quad \text{dense phase, } \alpha_0 > 0. \tag{9}$$

Since this action is local we can use the usual methods of conformal field theory. Such an action is known to describe a field theory with central charge

$$c = 1 - 24\alpha_0^2 = 1 - 3\frac{(\kappa-4)^2}{2\kappa}.$$

However, the saddle point of the action does not lie in the real fields and therefore the Bose field in the OPEs is not real so that it is no longer the coarse-grained height function.

4.1.1 Conformal transformations and boundary conditions

As an illustration, we consider the upper half plane (with punctures). In this case the curvature of the boundary is zero except at the point at infinity. By conformal invariance, the theory can be mapped onto any domain with a curved smooth boundary.

Since the field is Gaussian it consists of the holomorphic and the anti-holomorphic parts:

$$\varphi(z,\bar z) = \phi(z) + \bar\phi(\bar z),$$

whose correlation functions are

$$\langle\phi(z)\phi(z')\rangle = -\log(z-z'), \quad \langle\bar\phi(\bar z)\bar\phi(\bar z')\rangle = -\log(\bar z - \bar z'), \quad \langle\phi(z)\bar\phi(\bar z)\rangle = 0. \tag{10}$$

We will also use the dual boson field $\widetilde{\varphi}$ related to φ through Cauchy-Riemann conditions:

$$\widetilde{\varphi}(z,\bar{z}) = -i\phi(z) + i\bar{\phi}(\bar{z}). \tag{11}$$

Varying the action $\mathcal{A}$ with respect to the metric we find the components of the stress-energy tensor:

$$T(z) = -\frac{1}{2}(\partial\phi)^2 + i\sqrt{2}\alpha_0\partial^2\phi, \qquad \bar{T}(\bar{z}) = -\frac{1}{2}(\bar{\partial}\,\bar{\phi})^2 + i\sqrt{2}\alpha_0\bar{\partial}^2\bar{\phi}, \tag{12}$$

where $\partial = \partial/\partial z, \bar{\partial} = \partial/\partial\bar{z}$, and the products are normal ordered.

When the critical system is defined in the upper half plane the conformally invariant boundary condition is $T(x) = \bar{T}(x)$ for all $x \in \mathbb{R}$ [21]. For the stress-energy tensor (12) this condition readily yields $\phi(x) = \bar{\phi}(x)$, therefore $\varphi = \phi(z) + \phi(\bar{z})$ and $\widetilde{\varphi} = -i\phi(z) + i\phi(\bar{z})$ which is equivalent to the Neumann boundary condition:

$$\partial_y\varphi\big|_{y=0} = 0.$$

The Neumann condition implies that φ is a scalar which agrees with the meaning of the Bose field as height function.

The components (12) determine the transformation properties of the fields under a conformal transformation $z \to f(z)$:

$$\phi(z) \to \phi(f(z)) + i\sqrt{2}\alpha_0 \log f'(z), \tag{13}$$

which yields

$$\varphi(z,\bar{z}) \to \varphi(f(z), \overline{f(z)}) + i2\sqrt{2}\alpha_0 \log|f'(z)|.$$

This transformation is merely the transformation of the classical configuration of φ (the saddle point of (7)). Notice that although φ is not real, its real part does not transform so that the level lines

$$\mathrm{Re}\,\varphi(z,\bar{z}) = \mathrm{const.}$$

are conformally invariant fluctuating curves. This suggests to identify them with the continuous limit of critical curves i.e. the level lines of the coarse-grained height function.

4.1.2 Electric and magnetic operators

In the following we will see that critical curves in both dense and dilute phase are created by spinless vertex operators:

$$\mathcal{O}^{(e,m)}(z,\bar{z}) = e^{i\sqrt{2}e\varphi(z,\bar{z})}e^{-\sqrt{2}m\widetilde{\varphi}(z,\bar{z})}, \tag{14}$$

where we introduced "electric" and "magnetic" charges e and m. The boundary term in the action (9) is the insertion of an electric background charge $e = 2\alpha_0$ at a puncture in the upper half plane.

A vertex operator is a primary field and can be written as a product of the holomorphic and the antiholomorphic components $\mathcal{O}^{(e,m)}(z,\bar{z}) = V^{\alpha}(z)V^{\bar{\alpha}}(\bar{z})$, which are assumed to be normal ordered:

$$V^{\alpha}(z) = e^{i\sqrt{2}\alpha\phi(z)}, \qquad \bar{V}^{\bar{\alpha}}(\bar{z}) = e^{i\sqrt{2}\bar{\alpha}\bar{\phi}(\bar{z})}.$$

Here the holomorphic and antiholomorphic charges are

$$\alpha = e + m, \qquad \bar{\alpha} = e - m. \tag{15}$$

In the dense phase the Neumann boundary condition allows to write

$$\mathcal{O}^{(e,m)}(z,\bar{z}) = e^{i\sqrt{2}(e+m)\phi(z)}e^{i\sqrt{2}(e-m)\phi(\bar{z})}. \tag{16}$$

This can be interpreted as gluing together the holomorphic and the antiholomorphic sectors which is best understood in terms of image charges [21, 23, 24]. The operator (16) is regarded as a product of two holomorphic operators: one in the upper half plane and the other at the image point in the lower half plane. In the image the electric charge remains the same while the magnetic charge changes sign.

The holomorphic and the antiholomorphic weights of the vertex operators are found by applying the stress-energy tensor (12) to them:

$$h_{\alpha} = \alpha(\alpha - 2\alpha_0), \qquad \bar{h}_{\bar{\alpha}} = \bar{\alpha}(\bar{\alpha} - 2\alpha_0).$$

A given weight h corresponds to two charges: $\alpha_0 \pm \sqrt{\alpha_0^2 + h}$.

In terms of the electric and magnetic charges the weights read

$$h(e,m) = (e+m)(e+m-2\alpha_0), \quad \bar{h}(e,m) = (e-m)(e-m-2\alpha_0). \tag{17}$$

A vertex operator is spinless (meaning that $h = \bar{h}$) if either $\bar{\alpha} = \alpha$ or $\bar{\alpha} = 2\alpha_0 - \alpha$. In the first case the operator is purely electric ($m = 0$) and in

the second case it can have an arbitrary magnetic charge m but the electric charge should be $e = \alpha_0$.

We will also find it convenient to denote the vertex operators by their weights. Thus $\mathcal{O}_h(z, \bar{z})$ is a spinless vertex operator of weight h in the bulk and $\mathcal{O}_h(x)$ is a boundary vertex operator of weight h.

The propagators (10) lead to fusion by the addition of charges:

$$\lim_{z_1 \to z_2} V^{\alpha_1}(z_1) V^{\alpha_2}(z_2) = (z_1 - z_2)^{2\alpha_1\alpha_2} V^{\alpha_1+\alpha_2}(z_1) + \ldots. \tag{18}$$

In the literature it is customary to label holomorphic charges and weights by two numbers r, s according to

$$\begin{aligned} \alpha_{r,s} &= \alpha_0 - \frac{1}{2}(r\alpha_+ + s\alpha_-) = \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_-, \\ h_{r,s} &= \alpha_{r,s}(\alpha_{r,s} - 2\alpha_0) = \frac{(r\kappa - 4s)^2 - (\kappa - 4)^2}{16\kappa}. \end{aligned} \tag{19}$$

The primary field of weight $h_{r,s}$ is denoted $\psi_{r,s}$. We will use it as a shorthand notation without imposing any restrictions on r or s.

4.1.3 Curve-creating operators in the dense phase

In this section we determine the vertex operators which create critical curves.

On the lattice, the appearance of n critical curves emanating from a small region in the bulk results in the change in the height function H by $\pm\pi n$ as one makes a full circle around this region. This assumes that all of the curves are oriented the same way — inwardly or outwardly, — for otherwise they would be able to reconnect with each other. In the continuous limit $H \to \operatorname{Re}\varphi$ this represents a vortex configuration of the field φ. As the propagators (10) show, a vortex is produced by the insertion of a magnetic charge:

$$\begin{aligned} \mathcal{O}^{(e,0)}(z, \bar{z})\mathcal{O}^{(0,m)}(z', \bar{z}') &= \left(\frac{z - z'}{\bar{z} - \bar{z}'}\right)^{2em} \mathcal{O}^{(e+m)}(z')\mathcal{O}^{(e-m)}(\bar{z}') + \ldots \\ &= e^{4iem\arg(z-z')}\mathcal{O}^{(e+m)}(z')\mathcal{O}^{(e-m)}(\bar{z}') + \ldots \end{aligned}$$

This means that for $z \to z'$ the value of the field changes by $\pm 4\sqrt{2}\pi m$ as one goes around z' in full circle. Hence, a discontinuity line arises (Fig. 6). If this vortex corresponds to a star of n critical curves joining at the point z', the change in φ should be equal to $\pm n\pi\mathcal{R}$. The compactification of φ makes the discontinuity line invisible for an even number of curves. For an odd number of curves, a discontinuity line with the jump $\pi\mathcal{R}$ remains. One

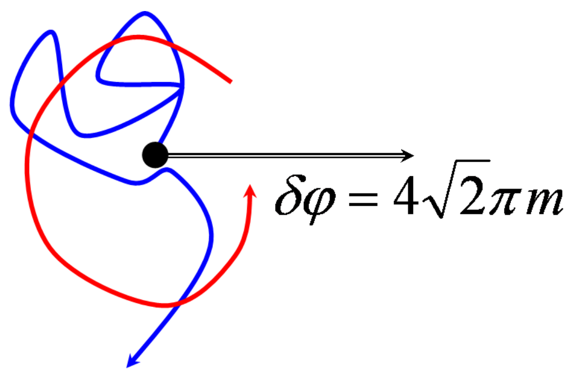

Figure 6: A bulk magnetic operator $\mathcal{O}^{(0,m)}$ creates a vortex configuration of the field φ. One critical curve is shown. The double line is the discontinuity line of the field.

can view it as a critical curve whose form is conditioned to be fixed so that it is removed from the ensemble of random curves.

We then find the magnetic charge of the bulk operator which creates n curves as

$$m = \pm\frac{\sqrt{2}}{8} n\mathcal{R} = \pm\frac{n}{2\sqrt{\kappa}} = \mp\frac{n}{4}\alpha_-,$$

where we used the value (6) of the compactification radius and the definition (8) of α_-. In order to be spinless (otherwise the operator would transform under rotations, giving non-trivial dependence on the winding number of curves), the bulk curve-creating operator should also have an electric charge α_0 thus acquiring the form

$$V^{\alpha_0+m}(z)V^{\alpha_0-m}(\bar{z}).$$

We have therefore found the possible holomorphic charges of the bulk curve-creating operator to be

$$\alpha = \alpha_0 \mp \frac{n}{4}\alpha_- = \alpha_{0,\pm n/2}$$

in the notation (19). These charges correspond to the same weight

$$h_{0,n/2} = \frac{4n^2 - (\kappa-4)^2}{16\kappa}.$$

The curve-creating operator itself is then denoted $\psi_{0,n/2}$. In particular, a single critical curve going through a point z (which is the same as $n = 2$

critical curves meeting at the point z) is created by the operator $\psi_{0,1}$ with the holomorphic weight $h_{0,1} = (8-\kappa)/16$. Note that this weight determines the fractal dimension of a critical curve as $2 - 2h_{0,1}$ [33].

The creation of curves on the boundary is considered in the same way. If n curves exit a boundary point the value of the height function changes by $\pm n\pi\mathcal{R}$ as one goes around this point in a semi-circle. On the real axis $[e^{\pm i\sqrt{2}\alpha\phi(x)}, \phi(y)] = \pm\sqrt{2}\pi\alpha\,\theta(x-y)e^{\pm i\sqrt{2}\alpha\phi(x)}$ where $\theta(x)$ is the step-function. This shows that the field value ϕ on the boundary changes by $\pm\sqrt{2}\pi\alpha$ in crossing the inserted vertex operator. Moreover, in presence of the Neumann boundary condition $\varphi(x) = 2\phi(x)$ on the real axis. The n curves are created, therefore, by the insertion of a boundary operator of the form $e^{\pm i\sqrt{2}\alpha_{1,n+1}\phi}$. The weight depends on the sign in the exponent. The lower weight corresponding to plus sign so that this operator is more relevant. This is then the curve-creating operator on the boundary denoted $\psi_{1,n+1}$. There also exist operators which condition a curve not to touch a certain part of the boundary [33, 34, 35, 36].

4.2 Dilute phase $\alpha_0 < 0$

Dilute critical curves occur, for example, in the $O(n)$ model at the critical temperature. They are known to be described by $g > 1$ ($\kappa < 4$). The central charge and the conformal weights of all curve-creating operators are obtained by continuation of the corresponding formulas from the dense phase ($\kappa > 4$). The arguments used to derive the action of field theory from the lattice in the dense phase cannot be straightforwardly extended to the dilute phase. Nevertheless, the description of the dilute phase is available via the electric-magnetic duality transformation $e \leftrightarrow m$ also called T-duality in the literature (see [5, 20, 37, 38]). Every correlation function of primary fields — and in particular curve-creating operators, — can be represented as the insertion of electric and magnetic charges into the partition function: $Z[g,e,m]$. The electric-magnetic duality means $Z[g,e,m] = Z[1/g,m,e]$.

The interchange between the electric and magnetic background charge means the following change of the boundary term in the action (9):

$$i\frac{\sqrt{2}\alpha_0}{2\pi}\int_{\partial D} dl\, K\varphi \to -\frac{\sqrt{2}\alpha_0}{2\pi}\int_{\partial D} dl\, K\widetilde{\varphi},$$

where $\widetilde{\varphi}$ is the Hilbert transform of φ on the boundary of the domain D, with the property $\partial_l\widetilde{\varphi}\big|_{\partial D} = -\partial_n\varphi\big|_{\partial D}$. The normal derivative is taken in the direction inside D. Once φ is harmonic $\widetilde{\varphi}$ coincides with the definition (11).

The electric-magnetic transformation makes φ a pseudo-scalar. As a result the action reads

$$\mathcal{A} = \frac{1}{8\pi}\int_D d^2x\,(\nabla\varphi)^2 - \frac{\sqrt{2}\alpha_0}{2\pi}\int_{\partial D} dl\,K\widetilde{\varphi}, \quad \text{dilute phase, } \alpha_0 < 0. \tag{20}$$

We note that in the dilute phase the action is real and its saddle point lies in the real fields. The connection of the field φ to the critical curves in the dilute phase will be discussed below.

For simplicity, we start with the theory in the upper half plane. The components of the stress-energy tensor now read (cf. (12))

$$T(z) = -\frac{1}{2}(\partial\phi)^2 + i\sqrt{2}\alpha_0\partial^2\phi, \qquad \bar{T}(\bar{z}) = -\frac{1}{2}(\bar{\partial}\,\bar{\phi})^2 - i\sqrt{2}\alpha_0\bar{\partial}^2\bar{\phi}. \tag{21}$$

The condition $T(x) = \bar{T}(x)$ now glues the holomorphic and antiholomorphic sectors on the real axis as $\phi(x) = -\bar{\phi}(x)$ which means $\varphi(z,\bar{z}) = \phi(z) - \phi(\bar{z})$, $\widetilde{\varphi} = -i\phi(z) - i\phi(\bar{z})$ and therefore the boundary condition on the real axis is Dirichlet:

$$\partial_x\varphi\big|_{y=0} = 0. \tag{22}$$

Unlike in the dense phase, $\bar{\phi}$ is complex conjugate of ϕ.

The holomorphic and antiholomorphic weights of $\mathcal{O}^{(e,m)}$ in the dilute phase are (cf. (17))

$$h(e,m) = (e+m)(e+m-2\alpha_0), \qquad \bar{h}(e,m) = (e-m)(e-m+2\alpha_0)$$

or

$$h = \alpha(\alpha - 2\alpha_0), \quad \bar{h} = \bar{\alpha}(\bar{\alpha} + 2\alpha_0)$$

in the notation (15). Since in the dilute phase $\alpha_0 < 0$, the charge which corresponds to a weight h and vanishes with h is

$$\alpha_h = \alpha_0 + \sqrt{\alpha_0^2 + h}, \quad \kappa \leqslant 4 \tag{23}$$

A spinless vertex operator in the dilute phase is either purely magnetic $e = 0$, or it has a magnetic charge $m = \alpha_0$ and an arbitrary electric charge.

The stress-energy tensor (21) shows that under a conformal transformation $z \to f(z)$

$$\varphi(z,\bar{z}) \to \varphi(f(z), \overline{f(z)}) - 2\sqrt{2}\alpha_0 \arg f'(z),$$

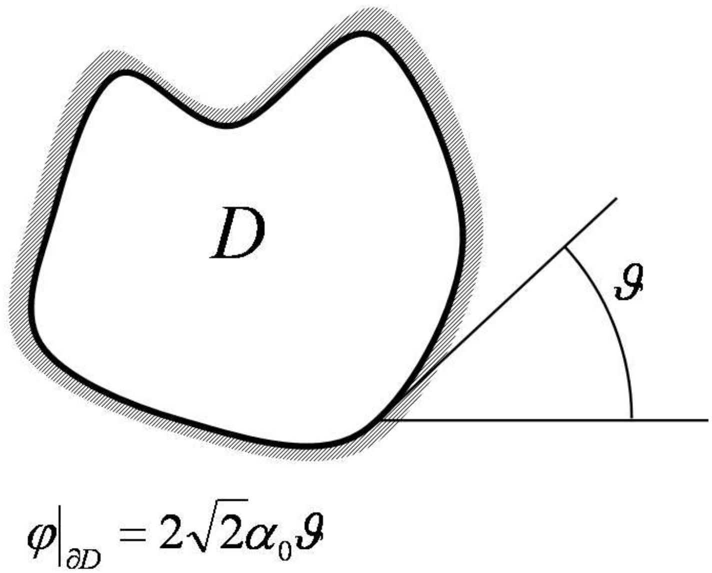

Figure 7: Boundary condition (25). $\vartheta = -\arg f'_D\big|_{\partial D}$.

which preserves the reality of φ. We observe that the flow lines of the vector field

$$e^{\frac{i}{2\sqrt{2}\alpha_0}\varphi} \tag{24}$$

are conformally invariant since (24) has conformal spin $h-\bar{h} = -1$. If φ were a smooth function the flow lines would repulse. This suggests identifying them with the critical curves in the dilute phase. The identification of flow lines (24) for the Gaussian field with Dirichlet boundary conditions with conformally invariant curves in SLE was done in [39].

In the dilute phase the boundary of the domain is a flow line. The boundary condition (22) transforms with the transformation of the domain. If $f_D(z)$ maps the domain D onto the upper half plane (a domain with a straight boundary) the boundary condition in D is (Fig. 7)

$$\varphi\big|_{\partial D} = -2\sqrt{2}\alpha_0 \arg f'_D\big|_{\partial D}. \tag{25}$$

At one point on the boundary φ undergoes a jump $4\sqrt{2}\alpha_0\pi$, which means that a magnetic charge $2\alpha_0$ is inserted there. In the upper half plane this magnetic charge is pushed out to infinity.

In the upper half plane the Dirichlet boundary condition allows us to write the vertex operator (14) as (cf. eq. (16))

$$\mathcal{O}^{(e,m)}(z,\bar{z}) = e^{i\sqrt{2}(e+m)\phi(z)} e^{i\sqrt{2}(m-e)\phi(\bar{z})},$$

which shows that the image of an electric charge changes sign and the image of the magnetic charge does not.

The conformal weights and holomorphic charges of the curve-creating operators are found by the continuation of the corresponding formulas from the dense phase. That is, the bulk operators giving rise to n critical dilute curves are $\psi_{0,n/2}$ and $\psi_{1,n+1}$, but with $\kappa < 4$ in the definitions (19) of their weights. We summarize the operators which create n curves in the bulk and on the boundary in both phases in the following table.

	n curves in bulk			n curves on boundary		
	r, s	e, m	explicit	r, s	α	explicit
Dense	$0, \frac{n}{2}$	$\alpha_0, \mp\frac{n}{4}\alpha_-$	$e^{i\sqrt{2}\alpha_0\varphi} e^{\pm\frac{\sqrt{2}n}{4}\alpha_-\widetilde{\varphi}}$	$1, n{+}1$	$-\frac{n}{2}\alpha_-$	$e^{-i\frac{n}{\sqrt{2}}\alpha_-\phi}$
Dilute	$0, \frac{n}{2}$	$\mp\frac{n}{4}\alpha_-, \alpha_0$	$e^{\mp i\frac{\sqrt{2}n}{4}\alpha_-\varphi} e^{-\sqrt{2}\alpha_0\widetilde{\varphi}}$	$1, n{+}1$	$-\frac{n}{2}\alpha_-$	$e^{-i\frac{n}{\sqrt{2}}\alpha_-\phi}$

4.3 Differential equations for the boundary curve-creating operator

Due to the conformal invariance any correlation function which includes a boundary curve-creating operator satisfies a differential equation which we derive below. The correlation functions then can be found as solutions of this equation. This applies to both the dense and the dilute phases.

The stress-energy tensor (12) or (21) is a field which generates transformations of primary fields under changes of geometry. The change of the upper half plane into that with a puncture at w_0 is produced by an infinitesimal transformation conformal everywhere except w_0:

$$z \to f(z) = z + \epsilon\Big(\frac{1}{z - w_0} + \frac{1}{z - \bar{w}_0}\Big), \qquad \epsilon \to 0.$$

This transformation affects correlation functions of primary fields in the usual way via the primary transformation law. A correlation function in the

new geometry is

$$\Big\langle \prod_{i=1}^{k} V^{\alpha_i}(z_i)\Big\rangle_{\mathbb{H}\backslash w_0} = \prod_{i=1}^{k}\Big(\frac{df^{-1}(z_i)}{dz_i}\Big)^{h_i}\Big\langle \prod_{i=1}^{k} V^{\alpha_i}(f^{-1}(z_i))\Big\rangle_{\mathbb{H}}$$
$$= \Big[1+\epsilon\sum_{i=1}^{k}\Big(\frac{h_i}{(z_i-w_0)^2}-\frac{1}{z_i-w_0}\partial_{z_i}+\{w_0\to\bar{w}_0\}\Big)\Big]\Big\langle \prod_{i=1}^{k} V^{\alpha_i}(z_i)\Big\rangle_{\mathbb{H}}.$$

The same change of the domain is by definition produced by the insertion of the stress-energy tensor $T(w_0)+\bar{T}(\bar{w}_0)$ into the correlation function (the conformal Ward identity).

We now fix w_0 to be a real number x. On the real axis $T(x)=\bar{T}(x)$ and the Ward identity reads

$$\langle T(x)\mathcal{O}(z_1,\dots z_k)\rangle_{\mathbb{H}} = \mathcal{L}_{-2}\langle\mathcal{O}(z_1,\dots z_k)\rangle_{\mathbb{H}},$$

where $\mathcal{O}(z_1,\dots z_k)=V^{\alpha_1}(z_1)\dots V^{\alpha_k}(z_k)$ and

$$\mathcal{L}_{-n}=\sum_{i=1}^{k}\Big(\frac{(n-1)h_i}{(z_i-x)^n}-\frac{1}{(z_i-x)^{n-1}}\partial_{z_i}\Big). \tag{26}$$

We remark that due to translational invariance the action of $\partial_x+\sum_{i=1}^{k}\partial_{z_i}$ on the correlation function is zero so that $\mathcal{L}_{-1}=\partial_x$.

The product of fields inside a correlation function is not normal-ordered. We can also insert the stress-energy tensor at the same point as one of the fields using normal ordering:

$$\langle :T(x)V^{\alpha}(x): O(z_1,\dots z_k)\rangle_{\mathbb{H}} = \mathcal{L}_{-2}\langle V^{\alpha}(x)O(z_1,\dots z_k)\rangle_{\mathbb{H}} \tag{27}$$

with the same definition of $\mathcal{L}_{-2}$. For special values of α this leads to a differential equation [2, 3].

Using the series expansion $V^{\alpha}(x)=\sum_{n=0}^{\infty}\frac{(i\sqrt{2}\alpha)^n}{n!}:\phi^n(x):$ and the Wick theorem we find the normal-ordered product

$$:\partial\phi(x)\partial\phi(x)V^{\alpha}(x): =:\partial\phi(x)\partial\phi(x)e^{i\sqrt{2}\alpha\phi(x)}: -i2\sqrt{2}\alpha :\partial^2\phi(x)e^{i\sqrt{2}\alpha\phi(x)}:,$$
$$:\partial^2\phi(x)V^{\alpha}(x): =:\partial^2\phi(x)e^{i\sqrt{2}\alpha\phi(x)}: .$$

The second term in the first line comes from only one of $\partial\phi(x)$ being contracted with the exponential. We then find

$$:T(x)V^{\alpha}(x):=:(-\frac{1}{2}\partial\phi(x)\partial\phi(x)+i\sqrt{2}(\alpha_0+\alpha)\partial^2\phi(x))e^{i\sqrt{2}\alpha\phi(x)}: .$$

On the other hand, a simple differentiation gives

$$\partial^2 V^\alpha(x) = :(-2\alpha^2 \partial\phi(x)\partial\phi(x) + i\sqrt{2}\alpha\partial^2\phi(x))e^{i\sqrt{2}\alpha\phi(x)}: .$$

These two expressions are the same up to a prefactor provided $4\alpha(\alpha+\alpha_0) = 1$. The two solutions of this equation are $\alpha_{1,2} = 1/\sqrt{\kappa}$ and $\alpha_{2,1} = -\sqrt{\kappa}/4$. They determine operators degenerate on level 2: $\psi_{1,2} \equiv e^{i\sqrt{2}\alpha_{1,2}\phi}$ and $\psi_{2,1} \equiv e^{i\sqrt{2}\alpha_{2,1}\phi}$. We then write

$$:\Big(\frac{\kappa}{4}\partial^2 - T(x)\Big)\psi_{1,2}(x): = 0, \qquad :\Big(\frac{4}{\kappa}\partial^2 - T(x)\Big)\psi_{2,1}(x): = 0.$$

These are operator equations, meaning that they can be inserted into any correlation function leading, for example, to

$$\langle :\Big(\frac{\kappa}{4}\partial_x^2 - T(x)\Big)\psi_{1,2}(x): O(z_1, \ldots z_k)\rangle_{\mathbb{H}} = 0.$$

If all fields are primary we combine this result with the Ward identity (27) and obtain the following differential equations:

$$\begin{aligned} (\frac{\kappa}{4}\mathcal{L}_{-1}^2 - \mathcal{L}_{-2})\langle\psi_{1,2}(x)O(z_1, \ldots z_k)\rangle_{\mathbb{H}} &= 0, \\ (\frac{4}{\kappa}\mathcal{L}_{-1}^2 - \mathcal{L}_{-2})\langle\psi_{2,1}(x)O(z_1, \ldots z_k)\rangle_{\mathbb{H}} &= 0. \end{aligned} \tag{28}$$

The first of these equations is relevant for calculations of correlation functions containing the boundary curve-creating operator $\psi_{1,2}$.

5 Stochastic (Schramm-) Loewner evolution (SLE)

In the previous sections correlation functions of boundary operators were interpreted as averaging over the shapes of critical curves. The shape of a curve fluctuates with some probability measure i.e. there is a certain statistical weight associated with each possible shape. The SLE approach [8] determines this measure. If a curve is conditioned to a certain shape objects of field theory explicitly depend on it.

In this section we briefly review SLE ([8]–[19]) and present the Bose field, its current, the vertex operators and the stress-energy tensor explicitly in terms of the shape of the curve. We restrict the discussion to the dilute case $\kappa \leqslant 4$.

5.1 Loewner equation

Consider the upper half-plane $\mathbb{H}$ in which an arbitrary non-self-intersecting curve γ starting from the origin is drawn. Let the curve be parameterized by $t \geqslant 0$ called "time". This setting defines a continuous family of slit domains each member of the family being the upper half plane with a part of γ up to some value t removed: $\mathbb{H}_t = \mathbb{H}\backslash\gamma_t$. Each slit domain $\mathbb{H}_t$ is topologically equivalent to the upper half plane since γ does not have self-intersections. We define the time-dependent conformal map $w_t(z)$ of the slit domain to the upper half plane: $\mathbb{H}_t \to \mathbb{H}$. To make the definition unique we specify the behavior of $w_t(z)$ at infinity:

$$w_t(z) = z - \xi_t + \frac{2t}{z} + \dots, \quad z \to \infty, \tag{29}$$

where ξ_t is a real function of time uniquely determined by the curve γ_t. The time-evolution of the map is given by the Loewner equation [40]

$$\partial_t w_t(z) = \frac{2}{w_t(z)} - \dot{\xi}_t, \quad w_t(z)\big|_{t=0} = z, \quad \xi_t\big|_{t=0} = 0. \tag{30}$$

At each z this equation is valid up to the time t when $w_t(z) = 0$. When this happens the point z no longer belongs to the slit domain (it has been swallowed by the curve). For a simple curve this happens when z is hit by the curve: $\gamma_t = z$. Hence, the Loewner function $w_t(z)$ maps the tip of the growing curve onto the origin.

It is obvious that in order for γ to be a continuous curve without branching the forcing ξ_t should be a continuous function. Actually, certain types of forcing produce curves with double points: γ can self-touch. On the event of self-touching a portion of the plane that has become encircled by γ, is removed from $\mathbb{H}$ together with the $[0, t]$ portion of the curve, thus ensuring that the topology of $\mathbb{H}_t$ does not change. The applicability range of the Loewner equation, therefore, extends beyond the originally planned description of one-dimensional objects, e.g. curves. In general, the objects that grow according to the Loewner equation (the unions of swallowed points) are called hulls.

The Loewner equation (30) with a specified forcing defines a curve or a hull in the upper half plane growing with time, the process known as the chordal Loewner evolution. Its basic property is that the final point of the evolution is $\lim_{t\to\infty} \gamma_t = \infty$ i.e. a point on the boundary of the domain $\mathbb{H}$. One can define the chordal Loewner evolution in any domain D topologically equivalent to $\mathbb{H}$ by a simple composition of maps. In this case

by the Riemann theorem there exists a conformal mapping $f(z) : D \to \mathbb{H}$, and $w_t(f(z))$ defines a growing curve γ_t in the domain D: it maps $D \backslash \gamma_t \to \mathbb{H}$. The final point of this evolution $f^{-1}(\infty)$ necessarily lies on the boundary of D.

5.2 Stochastic Loewner equation

The Loewner evolution can be used to describe any curve which starts from the origin and goes to infinity in the upper half-plane. We would like to apply it to a conformally invariant curve produced in CFT by the insertion of operators $\psi_{1,2}$ at 0 and at ∞. This dictates the necessary properties of the forcing ξ_t.

Since γ in a statistical system is a random curve the Loewner forcing ξ_t should be a stochastic process. This turns the Loewner equation (30) into a stochastic differential equation [41, 42] which has the form of a Langevin equation. In the rigorous sense it should be written in terms of stochastic differentials because ξ_t may (and will) be nowhere-differentiable:

$$dw_t(z) = w_{t+dt}(z) - w_t(z) = \frac{2dt}{w_t(z)} - d\xi_t, \qquad d\xi_t = \xi_{t+dt} - \xi_t. \tag{31}$$

The r.h.s. is evaluated at the time t (the Itô convention for stochastic analysis [41]). This remark is important when we deal with Markov processes: their behavior at any time is completely independent of their earlier behavior. In the following we will write stochastic differential equations with usual time-derivatives, as in (30), always understanding them in the Itô sense.

The properties that specify the driving function ξ_t are obtained from the following simple considerations. First of all, a critical curve γ in a translationally-invariant system has no special preference of going to the left or to the right. The ξ_t should have a distribution that is symmetric around zero leading to the requirement

$$\xi_t = -\xi_t \quad \text{in law.} \tag{32}$$

By this we mean the statistical identity: random quantities on both sides of the equation are taken from the same distribution but, in general, represent *different* realizations.

Next, the averaging over the ensemble of shapes of γ can be done in steps: first we can fix the shape of the curve up to a time t i.e. $\gamma[0, t]$, and average over the remaining part $\gamma[t, \infty]$. In this averaging $\gamma[0, t]$ is seen as a part of the domain boundary whose shape will be averaged later. With

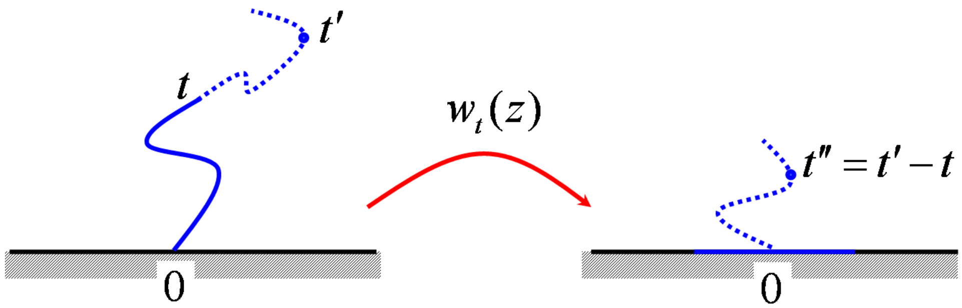

Figure 8: The image of the $[t, t']$ part of the curve on the left created by the mapping $w_t(z)$ is statistically identical to the $[0, t'']$ part of the same curve on the right.

a fixed $\gamma[0, t]$ the remaining part $\gamma[t, \infty]$ is a conformally invariant curve in the domain $\mathbb{H}_t$. By the conformal invariance we can map $\mathbb{H}_t$ onto $\mathbb{H}$ by $w_t(z)$ and the image of $\gamma[t, \infty]$ should be statistically identical to the original whole curve $\gamma[0, \infty]$ (see Fig. 8). In particular, the image of any segment $\gamma[t, t']$ (here $t' > t$) should be statistically identical to a segment $\gamma[0, t'']$ of the original curve:

$$\gamma[0, t''] = w_t(\gamma[t, t']) \quad \text{in law.}$$

The fact that we can perform the averaging in steps means that the r.h.s. and the l.h.s. are statistically independent. The application of $w_{t''}$ maps the curve in the l.h.s. into the real axis, and the same is done with the curve $\gamma[t, t']$ by the application of $w_{t'}$. Therefore, $w_{t'}(z) = w_{t''}(w_t(z))$ in law. At large z we can replace the Loewner maps with their asymptotic form (29) which allows us to deduce $t'' = t' - t$ and arrive to two equivalent statements

$$w_{t'}(z) = w_{t'-t}(w_t(z)) \quad \text{in law,} \tag{33}$$

$$\xi_{t'} = \xi_t + \xi_{t'-t} \quad \text{in law,} \tag{34}$$

where the r.h.s. and the l.h.s. are statistically independent in both formulas. Instead of eq. (33) we may equally well write

$$w_{t'}(z) = w_t(w_{t'-t}(z)) \quad \text{in law.} \tag{35}$$

The only continuous stochastic process ξ_t with stationary independent increments as in eq. (34) and the symmetry (32) is a driftless Brownian motion. Thus, it follows from the conformal invariance and continuity of the

curve that it should be described by stochastic Loewner evolution (SLE). It is defined by eq. (30) where the forcing ξ_t is a Brownian motion. The requirement of continuity is important here: it ensures that the curve does not branch and can be chronologically ordered in a unique way. Hence, the probability measure on a conformally invariant curve starting from the boundary is related to the Brownian motion via the Loewner equation. There is only one free parameter here: the strength of the noise κ defined by

$$\prec \dot{\xi}_t \dot{\xi}_{t'} \succ = \kappa \delta(t-t'),$$

where we use the notation $\prec \ldots \succ$ for averages over realizations of the Brownian motion ξ_t.

Different values of κ must give rise to conformally invariant curves in different statistical systems. Actually, the curve γ is simple i.e. never touches the boundary of the domain (or itself, by conformal invariance), if and only if $\kappa \leqslant 4$ [31] as in Fig. 4. This is consistent with parametrization of systems with dilute critical curves by $\kappa < 4$ in Section 4. At $\kappa > 4$ the Loewner transformation $w_t(z)$ maps the external perimeter of γ_t onto the real axis.

Next we turn to the formula (35) which we rewrite using the sign $\circ$ for composition of maps as $w_t^{-1} \circ w_{t'} = w_{t'-t}$ in law. Note that $w_t^{-1} \circ w_{t'}$ maps $\mathbb{H}_{t'}$ onto $\mathbb{H}_t$. Consider now an infinitesimal time increment: $dt = t' - t$, for which we have $w_t^{-1} \circ w_{t+dt} = w_{dt}$ in law. The infinitesimal Loewner map in the r.h.s. is given by (see eq. (31))

$$w_{dt}(z) = z + \frac{2dt}{z} - d\xi_t. \tag{36}$$

Then the statistically equivalent infinitesimal map $w_t^{-1} \circ w_{t+dt}$ from $\mathbb{H}_{t+dt}$ to $\mathbb{H}_t$ is given by the same equation:

$$w_t^{-1}(w_{t+dt}(z)) = z + \frac{2dt}{z} - d\xi_t. \tag{37}$$

We stress, however, that the equivalence of these maps is only statistical and the Brownian motions ξ_t in the two formulas (36, 37) are two *different* realizations drawn from the same statistical ensemble. Treating them as the same function would lead to an inconsistency because no function ξ_t (apart from a linear function at) satisfies the requirement (34) in the exact, not statistical, sense.

5.3 Martingales and correlation functions

If a stochastic process $\mathcal{O}(t)$ is a martingale, its average does not depend on time. Below we explain how a martingale $\mathcal{O}(t)$ generated in SLE corresponds

to a CFT holomorphic operator $\mathcal{O}$ such that [10]

$$\prec\mathcal{O}(t)\succ = C_{\mathcal{O}}(\alpha_0)\frac{\langle\psi_{1,2}(0)\mathcal{O}\psi_{1,2}(\infty)\Psi(\infty)\rangle_{\mathbb{H}}}{\langle\psi_{1,2}(0)\psi_{1,2}(\infty)\rangle_{\mathbb{H}}}. \tag{38}$$

The identification is up to a constant $C_{\mathcal{O}}$ which depends on α_0. We do not discuss it here.

Comparison with the two-step averaging (2) in which, as we now know, the curve-creating operator is $\mathcal{B} = \psi_{1,2}$, suggests the identification

$$\mathcal{O}(t) \sim \langle\mathcal{O}\Psi(\infty)\rangle_{\gamma_t}.$$

We will demonstrate the correspondence in the case of primary fields.

We will further explore the relation between CFT and SLE and show that at $t \to \infty$ the SLE fields $\mathcal{O}(t)$ satisfy the operator algebra of their counterparts in CFT. This is to be expected. The size of γ_t introduces a length scale $\sim \sqrt{t}$. When it diverges at $t \to \infty$, the local fields in SLE fluctuate in a scale-invariant way. In the following discussion we consider only the dilute phase $\kappa \leqslant 4$.

5.3.1 Primary fields in SLE

The Loewner equation can be considered in any domain. The fields related to the evolving curve transform under the conformal map f which maps one domain onto another. We call an SLE primary field a field which evolves under the Loewner equation and transforms as $\mathcal{O}_h(z) \to f'^h(z)\mathcal{O}_h(f(z))$ under a change of the domain. In particular, we can consider a pair of domains $\mathbb{H}$ and $\mathbb{H}_t$ related by conformal map $w_t(z)$ in which case

$$\mathcal{O}_h(z,t) = w_t'(z)^h\mathcal{O}_h(w_t(z),0). \tag{39}$$

Alternatively, we can use the pair of domains $\mathbb{H}_t$ and $\mathbb{H}_{t+dt}$ related by an infinitesimal conformal transformation $w_t^{-1}(w_{t+dt}(z))$ in which case

$$\mathcal{O}_h(z,t+dt) = \big(\partial_z w_t^{-1}(w_{t+dt}(z))\big)^h\mathcal{O}_h(w_t^{-1}(w_{t+dt}(z)),t).$$

Now we use the stochastic nature of SLE: the statistical identity of $w_t^{-1} \circ w_{t+dt}$ and w_{dt} and the explicit expression (37) enables us to write

$$\mathcal{O}_h(z,t+dt) = w_{dt}'(z)^h\mathcal{O}_h(w_{dt}(z),t) = \Big(1-\frac{2dt}{z^2}\Big)^h\mathcal{O}_h\Big(z+\frac{2dt}{z}-d\xi_t,t\Big).$$

Expanding the r.h.s. to the first order in dt and to the second order in $d\xi_t$ produces the evolution equation for $\mathcal{O}_h(z,t)$:

$$\partial_t \mathcal{O}_h(z,t) = \Big(\frac{\kappa}{2}\mathcal{L}_{-1}^2 - 2\mathcal{L}_{-2} + \dot{\xi}\mathcal{L}_{-1}\Big)\mathcal{O}_h(z,t), \tag{40}$$

where we used the notation (26) with $k=1$ and the position of the curve-creating operator x set to 0. The derivation is extended to a product of k primary fields, when operators $\mathcal{L}_{-1}$, $\mathcal{L}_{-2}$ act on k variables.

The two ways of representing the stochastic evolution of $\mathcal{O}_h(z,t)$ in the time-derivative of eq. (39) and in eq. (40) are equivalent but cannot be obtained from each other by a simple substitution since they contain different realizations of Brownian motion.

Since we understand stochastic equations in the Itô sense, the differential of the Brownian motion $\xi_{t+dt} - \xi_t$ in the last term of eq. (40) is multiplied by $\partial_z \mathcal{O}_h(z,t)$ evaluated at time t. Therefore, by the Markov property of the Brownian motion this term vanishes upon taking the average. The expectation value of a primary field then satisfies the Feynman-Kac partial differential equation

$$\partial_t \prec\!\mathcal{O}_h(z,t)\!\succ = \Big(\frac{\kappa}{2}\mathcal{L}_{-1}^2 - 2\mathcal{L}_{-2}\Big) \prec\!\mathcal{O}_h(z,t)\!\succ , \tag{41}$$

and similarly for a product of k fields. The evolution equation for a primary field is obtained solely from its transformation law under an infinitesimal conformal map. The same method allows to obtain the evolution equation for any field with a specified transformation law, not necessarily primary.

An example of a primary field is

$$\mathcal{O}_h(z,t) = t^{\alpha_{1,2}\alpha_h} w_t'(z)^h, \tag{42}$$

using the notation (19) and (23). As follows from the scaling law of SLE:

$$w_t(z) = \frac{1}{a} w_{a^2 t}(za), \quad \text{in law,}$$

the solution of the equation (41) for a real $z = x$ in this case must have the form $t^{\alpha_{1,2}\alpha_h} F(x/\sqrt{t})$ with the finite asymptote $x^{2\alpha_{1,2}\alpha_h}$ at $t \to \infty$. In this limit the fluctuations of $\mathcal{O}_h$ become scale-invariant and its average corresponds to a correlation function in CFT. There local fields have a singular short distance expansion. We now comment on how it arises in the SLE approach.

Consider the product of two boundary fields $t^{\alpha_{1,2}(\alpha_{h_1}+\alpha_{h_2})} w_t'(x_1)^{h_1} w_t'(x_2)^{h_2}$. Taking $x_1 = x_2$ we obtain $t^{\alpha_{1,2}(\alpha_{h_1}+\alpha_{h_2})} w_t'(x_1)^{h_1+h_2}$, which diverges at $t \to$

∞ due to convexity of α_h. The properly normalized field $t^{\alpha_{1,2}\alpha_{(h_1+h_2)}} w_t'(x_1)^{h_1+h_2}$ is different. This is an example of the short distance singularity. It closely parallels the origin of such singularities in field theory.

Since $\partial_t \prec \mathcal{O}_h(z,t) \succ$ vanishes at $t \to \infty$, the Feynman-Kac equation (41) in this limit becomes identical to eq. (28). Therefore, the SLE average $\prec\mathcal{O}_h(z,\infty)\succ$ and the CFT correlation function $\langle \psi_{1,2}(0) V_{\alpha_h}(z) \Psi(\infty) \rangle_{\mathbb{H}}$ obey the same equation. However, this fact alone does not allow to identify them as the following simple remark shows. The CFT correlation function is a holomorphic function $z^{2\alpha_{1,2}\alpha_h}$ with a branch point at the origin. It acquires a phase in going from a real z to $-z$. However, the SLE field $t^{\alpha_{1,2}\alpha_h} w_t'(z)^h$ does not since w_t' is real on the real axis.

The proper choice of the primary field of weight h is[2]

$$V^{\alpha_h}(z,t) = w_t(z)^{2\alpha_{1,2}\alpha_h} w_t'(z)^h. \tag{43}$$

Its expectation value is holomorphic and can be identified with the holomorphic CFT correlation function $\langle \psi_{1,2}(0) V^{\alpha_h}(z) \Psi(\infty) \rangle = z^{2\alpha_{1,2}\alpha_h}$. Indeed, its initial value is $V^{\alpha_h}(z,t)\big|_{t=0} = z^{2\alpha_{1,2}\alpha_h}$ and as we will show momentarily, its average does not depend on time.

Using the Loewner equation and the Itô formula for a product of two stochastic processes X_t and Y_t:

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t \tag{44}$$

with $X_t = w_t(z)^{2\alpha_{1,2}\alpha_h}$ and $Y_t = w_t'(z)^h$ we find

$$\partial_t V^{\alpha_h}(z,t) = -\frac{2\alpha_{1,2}\alpha_h}{w_t(z)} V^{\alpha_h}(z,t)\dot{\xi}_t,$$

which yields $\partial_t \prec V^{\alpha_h}(z,t) \succ = 0$ i.e. $V^{\alpha_h}(z,t) = w_t(z)^{2\alpha_{1,2}\alpha_h} w_t'(z)^h$ is a martingale. We note that $\alpha_{1,2} = 1/\sqrt{\kappa}$ and thus $\alpha_{1,2}\xi_t$ is a Brownian motion B_t with variance 1.

We conclude that $\prec V^{\alpha_h}(z,t)\succ$ satisfies the stationary equation which is identical to (28):

$$\left(\frac{\kappa}{4}\mathcal{L}_{-1}^2 - \mathcal{L}_{-2}\right) \prec V^{\alpha_h}(z,t)\succ = 0.$$

at any time and not only at $t \to \infty$. The same equation holds for a string of k primary fields.

In the remainder of this section we will further illustrate the correspondence between SLE and Bose field.

[2] This formula is understood up to a multiplicative constant related to the constant $C_{\mathcal{O}}(\alpha_0)$ in (38).

5.3.2 Bose field in SLE

We now will discuss the SLE Bose field $\phi(z,t)$, a martingale which corresponds to the holomorphic part of the Bose field discussed in Sec. 4.2:

$$\phi(z,t) \sim \langle \psi_{1,2}(0)\phi(z)\Psi(\infty)\rangle_{\gamma_t}.$$

It follows that $\phi(z,t)$ has the transformation law (13) and is a martingale. This fixes it to be [39]

$$\phi(z,t) = i\sqrt{2}\alpha_0 \log w_t'(z) - \frac{i\mathcal{R}}{2}\log w_t(z), \quad \mathcal{R} = \sqrt{\frac{8}{\kappa}},$$

up to an additive constant (cf. footnote 2). Indeed, it follows from the Loewner equation that $d\phi(z,t) = \frac{i\mathcal{R}}{w_t(z)}d\xi_t$ and therefore the average of $\phi(z,t)$ equals its initial value $\prec\phi(z,t)\succ = -\frac{i\mathcal{R}}{2}\log z$. The latter is just the holomorphic part of the solution of the Laplace equation with a step-like boundary condition $\varphi(x<0) = \pi\mathcal{R}$, $\varphi(x>0) = 0$.

The real field $\varphi(z,\bar z,t) = \phi(z,t) + \overline{\phi(z,t)}$ is compactified with radius $\mathcal{R}$. If we choose a point z_γ to lie on the slit γ the value of φ is given by the direction ϑ of the tangential vector[3]:

$$\varphi_L(z_\gamma) = 2\sqrt{2}\alpha_0\vartheta + \pi\mathcal{R}, \quad \varphi_R(z_\gamma) = 2\sqrt{2}\alpha_0(\vartheta - \pi), \tag{45}$$

where the additive constants correspond to the two sides of the slit (Fig. 9). The Bose field then measures the winding angle of γ. The tangential vector of γ is $e^{i\varphi_R/2\sqrt{2}\alpha_0}$, as in (24). The identification of the Bose field with an angle emphasizes that in the dilute phase it is a pseudo-scalar. The field fluctuates with γ but its jump across γ is fixed:

$$(\varphi_L - \varphi_R)\big|_\gamma = \pi(\mathcal{R} + 2\sqrt{2}\alpha_0) = \sqrt{\frac{\kappa}{2}}\pi.$$

On the real axis φ obeys the Dirichlet boundary condition: $\varphi(x,t) = 0$ for $x>0$ and $\varphi(x,t) = \pi\mathcal{R}$ for $x<0$.

The case $\kappa = 4$ (i.e. $\alpha_0 = 0$) is special: the values of the field on γ_t then are fixed to be 0 and $\pi\mathcal{R}$ as on the real axis.

The evolution equation for the Bose field follows from its transformation law and the Loewner equation similarly to the previous section. It is given by

$$\partial_t\phi(z,t) = \Big(\frac{\kappa}{2}\mathcal{L}_{-1}^2 - 2\mathcal{L}_{-2} + \dot\xi\mathcal{L}_{-1}\Big)\phi(z,t) - \frac{i2\sqrt{2}\alpha_0}{z^2}, \tag{46}$$

[3] In fact, γ is a fractal curve. It is understood that ϑ is measured not on γ but nearby.

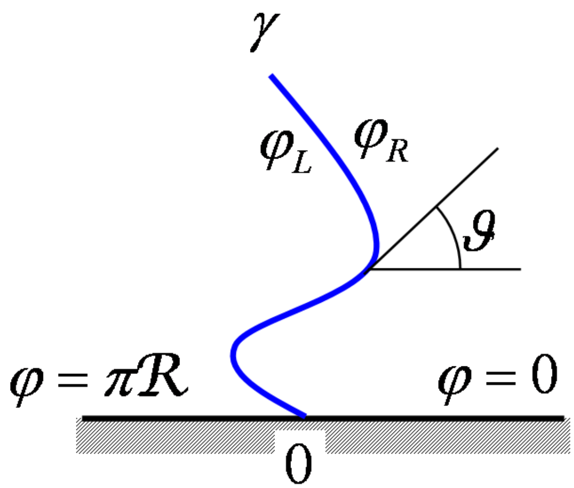

Figure 9: The SLE field φ on γ measures the winding angle ϑ (see eq. (45)). The tangential component of the current $\partial_l \varphi$ on γ measures the curvature.

with $h = 0$ in the definitions (26) of $\mathcal{L}$'s. Since $\phi(z,t)$ is a martingale its average $\prec\phi(z,t)\succ = -\frac{i\mathcal{R}}{2}\log z$ solves the equation

$$\left(\frac{\kappa}{4}\mathcal{L}_{-1}^2 - \mathcal{L}_{-2}\right)\prec\phi(z,t)\succ \; -\frac{i\sqrt{2}\alpha_0}{z^2} = 0. \tag{47}$$

This is the same equation as $\langle\psi_{1,2}(0)\phi(z)\Psi(\infty)\rangle$ satisfies.

We now comment on how the operator algebra of the Bose field arises in the SLE approach. With the help of the Itô formula (44) we find from the Loewner equation the evolution of the product of two Bose fields:

$$d(\phi(z_1,t)\phi(z_2,t)) = d\phi(z_1,t)\phi(z_2,t) + \phi(z_1,t)d\phi(z_2,t) + d\phi(z_1,t)d\phi(z_2,t).$$

One can check that the last term (drift) is

$$d\phi(z_1,t)d\phi(z_2,t) = -\frac{2dt}{w_t(z_1)w_t(z_2)} = d\log(w_t(z_1) - w_t(z_2)),$$

and therefore $\phi(z_1,t)\phi(z_2,t)$ is not a martingale. The martingale reads

$$\phi(z_1,t)\phi(z_2,t) - \log(w_t(z_1) - w_t(z_2)). \tag{48}$$

This process is identified with the product of operators in CFT. The subtracted term here is non-local. However, at $t \to \infty$ it vanishes — a manifestation of the fact that only in this limit local fields of SLE fluctuate as local fields of CFT.

In the limit $z_1 \to z_2$ the subtracted term becomes $\log(w_t(z_1) - w_t(z_2)) = \log w_t'(z_1) + \log(z_1 - z_2)$. We define the normal-ordered product by subtracting the logarithmic divergence from (48):

$$:\phi^2(z,t):= \phi^2(z,t) - \log w_t'(z).$$

The Itô formula (44) applied to a product of more than two bosons becomes the Wick theorem. The normal-ordered exponential is found up to a multiplicative constant as

$$:e^{i\sqrt{2}\alpha_h \phi_t(z)}:= w_t'(z)^{\alpha_h^2} e^{i\sqrt{2}\alpha_h \phi_t(z)} = w_t^{2\alpha_{1,2}\alpha_h}(z) w_t'(z)^h,$$

which is just the definition (43) of $V^{\alpha_h}(z,t)$.

The holomorphic current of the Bose field is

$$j(z,t) = \partial\phi(z,t) = i\sqrt{2}\alpha_0 \frac{w_t''(z)}{w_t'(z)} - i\frac{\mathcal{R}}{2}\frac{w_t'(z)}{w_t(z)}.$$

Since φ is a pseudo-scalar, the current is an axial vector. On the curve γ its tangential component is proportional to the geodesic curvature of γ:

$$j_l\big|_\gamma = 2\sqrt{2}\alpha_0 K_\gamma.$$

This is consistent with the action (20). At the tip of the curve γ_t the current has a vortex with a charge proportional to α_0.

The evolution of $j(z,t)$ and the equation for $\prec j(z,t) \succ$ are found by differentiating eqq. (46,47). Differentiating (48) and taking $z_1 \to z_2$ we get the normal ordered square of the current

$$:j^2(z,t):= j^2(z,t) - \frac{1}{6}\{w_t, z\},$$

where $\{w,z\}$ is the Schwarz derivative:

$$\{w,z\} = \frac{w'''(z)}{w'(z)} - \frac{3}{2}\Big(\frac{w''(z)}{w'(z)}\Big)^2.$$

Finally, we can define the holomorphic stress-energy tensor in SLE as a martingale with the proper transformation law:

$$z \to f(z): \quad T(z,t) \to f'(z)^2 T(f(z),t) + \frac{c}{12}\{f,z\},$$

with $c = 1 - 24\alpha_0^2$. This fixes it to be

$$T(z,t) = -\frac{1}{2} :\partial\phi(z,t)\partial\phi(z,t): + i\sqrt{2}\alpha_0 \partial^2\phi(z,t),$$

which explicitly reads

$$T(z,t) = \frac{c}{12}\{w_t, z\} + h_{1,2}\left(\frac{w_t'(z)}{w_t(z)}\right)^2, \quad h_{1,2} = \frac{6-\kappa}{2\kappa}.$$

The combination $T(z,t)dz^2$ can be used to determine the probability that γ intersects a segment of size dz centered at z [43, 44].

The averages of the current and the stress-energy tensor are

$$\prec j(z,t) \succ = -\frac{i\sqrt{2}\alpha_{1,2}}{z}, \quad \prec T(z,t) \succ = \frac{h_{1,2}}{z^2}$$

We note that the current and the stress-energy tensor measure the charge and the weight of the curve-creating operator $\psi_{1,2}$.

We thus have seen that the stochastic Loewner evolution allows to write the objects of field theory explicitly in terms of the shape of the curve. These objects can be used to find distributions of various characteristics of curves. As an application we compute in the next section the harmonic measure of critical curves for $\kappa \leqslant 4$.

6 Harmonic measure of critical curves

In this section we apply the results of the previous sections to compute the multifractal spectrum of the harmonic measure of dilute ($\kappa \leqslant 4$) critical curves. In the dense phase $\kappa > 4$ these results apply to external perimeters of critical curves.

Harmonic measure quantifies geometry of complicated plane domains [45, 46]. We start with basic definitions.

6.1 Harmonic measure of critical curves

Consider a closed simple curve γ. One can imagine that it is made of a conducting material and carries a total electric charge one. The harmonic measure of any part of γ is defined as the electric charge of this part. In what follows we will pick a point of interest z_0 on the curve and consider a disc of a small radius $r \ll R$ centered at z_0. It surrounds a small part of γ and we define $\mu(z_0, r)$ to be the harmonic measure of this part i.e. the electric charge in it.

Then we consider the moments

$$M_h = \sum_{i=1}^{N} \mu(z_i, r)^h, \tag{49}$$

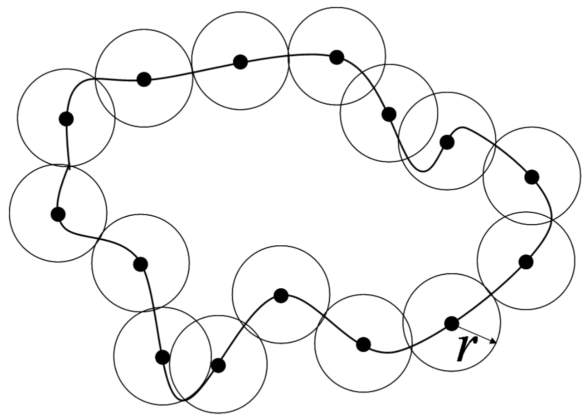

Figure 10: A curve covered by discs.

where h is a real power and N is the number of discs needed to cover γ as in Fig. 10. As the radius r gets smaller and the number of discs N gets larger, these moments scale as

$$M_h \sim \left(\frac{r}{L}\right)^{\tau(h)}, \quad \frac{r}{L} \to 0,$$

where L is the typical size of the curve.

The function $\tau(h)$ is called the multifractal spectrum of the curve γ. This function encodes a lot of information of the curve γ. It also has some simple properties. First of all, since all $0 < \mu(z_i, r) \leqslant 1$, the moments M_h are well defined for any real h and the function $\tau(h)$ is non-decreasing: $\tau(h) \leqslant \tau(h')$ for any $h < h'$. Secondly, if $h = 1$ the sum in (49) is equal to the total charge of the cluster and therefore does not scale with r, producing the normalization condition $\tau(1) = 0$. Thirdly, if we set $h = 0$, M_0 is simply the number N of discs of radius r necessary to cover the curve γ so that by definition the fractal (Hausdorff) dimension of γ is $d_f = -\tau(0)$.

If the curve γ were smooth we would have a simple relation $\tau(h) = h - 1$. For a fractal curve one defines the anomalous exponents $\delta(h)$ by

$$\tau(h) = h - 1 + \delta(h).$$

Also, the generalized multifractal dimensions of a fractal curve γ are defined as $D(h) = \tau(h)/(h-1)$ (so that $D(0) = d_f$). A theorem [47, 48] states that $D(1) = \tau'(1) = 1$.

If the curve γ is random, so are its moments M_h, and we can average them over the ensemble of curves. It is natural to assume that the summation

over the points z_i in eq. (49) is equivalent to the ensemble average. Hence we can write

$$\prec M_h \succ = N \prec \mu(r)^h \succ \sim \left(\frac{r}{L}\right)^{\tau(0)} \prec \mu(r)^h \succ ,$$

where now the harmonic measure $\mu(r)$ is measured at any point z on γ and $\prec \ldots \succ$ indicates the ensemble averaging. We define the local multifractal exponent $\widetilde{\tau}(h) = \tau(h) - \tau(0)$ at a point z by

$$\prec \mu(r)^h \succ \ \sim \left(\frac{r}{L}\right)^{\widetilde{\tau}(h)}. \tag{50}$$

The ensemble of critical curves is suitable for multifractal analysis. There are a few generalizations of the simple closed curve considered above. First of all, the curve γ need not be closed or stay away from the system boundaries. If γ touches a boundary we can supplement it with its mirror image across this boundary. The definition of $\mu(z, r)$ can be naturally extended to the cases when z is the endpoint of n critical curves on the boundary or in the bulk. If n is even, the latter case can be also seen as $n/2$ critical curves passing through z. In particular, $n = 2$ corresponds to the situation of a single curve in the bulk considered above.

When z is the endpoint of n critical curves on the boundary or in the bulk we define the corresponding scaling exponents similarly to eq. (50):

$$\prec \mu(z, r)^h \succ \ \sim r^{h+\Delta^{(n)}(h)}, \qquad \prec \mu(z, r)^h \succ \ \sim r^{h+\Delta^{(n)}_{\mathrm{bulk}}(h)}. \tag{51}$$

In the case of a single curve we will drop the superscript, for example, $\Delta(h) \equiv \Delta^{(1)}(h)$.

These exponents were first obtained by means of quantum gravity in [4, 5]. For a critical system parameterized by κ the results read

$$\Delta(h) = \frac{\kappa - 4 + \sqrt{(\kappa-4)^2 + 16\kappa h}}{2\kappa} = \frac{\sqrt{1-c+24h} - \sqrt{1-c}}{\sqrt{25-c} - \sqrt{1-c}}, \tag{52}$$

$$\Delta^{(n)}(h) = n\Delta(h), \tag{53}$$

$$\Delta^{(n)}_{\mathrm{bulk}}(h) = -\frac{h}{2} + \left(\frac{1}{16} + \frac{n-1}{4\kappa}\right)\left(\kappa - 4 + \sqrt{(\kappa-4)^2 + 16\kappa h}\right). \tag{54}$$

Remarkably, $\Delta(h)$ is the gravitationally dressed dimension h as given by the KPZ formula of 2D quantum gravity [7]. Connections of this formula to SLE were discussed in [35]. Starting in the next section we will show how to obtain these exponents formulating the $c \leqslant 1$ CFT via the Bose field, where they are also written in a more transparent way.

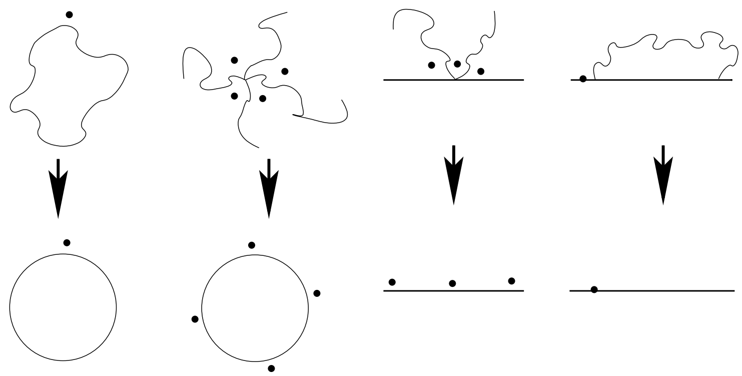

Figure 11: The uniformizing conformal maps for various cases considered. The dots denote that points where the electric field is measured.

6.2 Harmonic measure and uniformizing conformal maps

Let us consider a conformal map $w(z)$ of the exterior of γ to a standard domain. Usually we will choose the upper half plane but sometimes the exterior of a unit circle is more convenient. We choose the map so that the point of interest z on the curve is mapped onto itself, choose it to be the origin $z = 0$ and normalize the conformal map so that $w(\infty) = \infty$ and $w'(\infty) = \text{const}$. In the case of the upper half plane we choose $w'(\infty) = 1$ as we did in SLE (see Section 5), while for the unit circle $w'(\infty) = \rho^{-1}$ where ρ is the conformal radius of γ.

The scaling of $w(z)$ near the origin is directly related to that of the harmonic measure. Indeed, since $\mu(0, r)$ is the charge inside the disc of radius r by the Gauss theorem it is equal to the flux of the electric field through the boundary of this disc. This, in turn, should scale as the circumference of the disc times a typical absolute value of the electric field at a distance r from the origin i.e. $|w'(r)|$. This leads to a scaling relation

$$\mu(0, r) \sim r|w'(r)|,$$

which allows to rewrite the definitions (51) as

$$\prec\!|w'(r)|^h\!\succ\ \sim r^{\Delta^{(n)}(h)}, \qquad\qquad \prec\!|w'(r)|^h\!\succ\ \sim r^{\Delta^{(n)}_{\text{bulk}}(h)}.$$

The relation between the scaling of the harmonic measure and the derivative of a uniformizing map allows further generalizations. Namely, we can measure the electric field in more than one point. Consider n critical curves

emanating from the origin. Close to the origin the curves divide the plane into n sectors if the origin lies in the bulk and $n+1$ sectors if it lies on the boundary. Then we can study objects like

$$\prec\prod_i |w'(z_i)|^{h_i}\succ ,$$

where no two z_i's lie in the same sector. The case when the electric field is not measured in some sectors is done by setting $h_i = 0$ in them. We will see how to express these quantities as CFT correlation functions. In the case when z_i's are all at the distance r from the origin these averages scale as $r^{\Delta^{(n)}(h_1,\dots h_{n+1})}$ and $r^{\Delta^{(n)}_{\text{bulk}}(h_1,\dots h_n)}$ with the higher multifractal exponents related to the multifractal exponents (52 – 54) as [5]

$$\Delta^{(n)}(h_1,\dots h_{n+1}) = \sum_{i=1}^{n+1}\Delta^{(n)}(h_i) + \frac{\kappa}{2}\sum_{i<j}^{n+1}\Delta(h_i)\Delta(h_j), \tag{55}$$

$$\Delta^{(n)}_{\text{bulk}}(h_1,\dots h_n) = \sum_{i=1}^{n}\Delta^{(n)}_{\text{bulk}}(h_i) + \frac{\kappa}{4}\sum_{i<j}^{n}\Delta(h_i)\Delta(h_j). \tag{56}$$

In the case when z_i's are independent variables the answer is more complicated. However, in the case of a single curve at the boundary it reads:

$$\prec|w'(x_1)|^{h_1}|w'(x_2)|^{h_2}\succ \sim |x_1|^{\Delta(h_1)}|x_2|^{\Delta(h_2)}|x_1-x_2|^{\Delta(h_1,h_2)-\Delta(h_1)-\Delta(h_2)}. \tag{57}$$

6.3 Calculation of multifractal spectrum

Here we obtain formulas (52 – 54, 55, 56). We will consider in detail the simplest case of harmonic measure of a single curve near the boundary. The other cases contain a few technical differences. The calculation is based on the two-step averaging of Sec. 3.

6.3.1 Boundary multifractal exponent from CFT

All averages over the ensemble of critical curves are some correlation functions of the Bose field. Averages of w'^h are related to correlation functions of primary fields $\mathcal{O}_h$ in the presence of curve-creating operators. We start with the case of a single curve starting from the boundary, when this operator is $\psi_{1,2}$ and find the boundary multifractal exponent $\Delta(h)$.

We use the two-step averaging result (2) where we write $\mathcal{B} \equiv \psi_{1,2}$ and choose $\mathcal{O}$ to be a boundary vertex operator $\mathcal{O}_h(r) = e^{i\sqrt{2}\alpha_h \phi(r)}$. We uniformize the exterior of γ by mapping it to the upper half plane. Being a primary operator of weight h, $\mathcal{O}_h(r)$ transforms as $\mathcal{O}_h \to w'(r)^h \mathcal{O}_h(w(r))$ while $\Psi(\infty)$ does not transform because of the normalization of $w(z)$ at infinity. The transformation relates the correlation function in the exterior of γ to a correlation function in the upper half plane:

$$\langle \mathcal{O}_h(r)\Psi(\infty)\rangle_\gamma = w'(r)^h \langle \mathcal{O}_h(w(r))\Psi(\infty)\rangle_{\mathbb{H}}. \tag{58}$$

Note that the correlation function in the r.h.s. has no singular dependence on r, hence, as long as we stay in the limit $r \ll |L|$ it is a constant prefactor. We then obtain a scaling relation[4]

$$\prec w'(r)^h \succ \sim \langle \mathcal{O}_h(r)\psi_{1,2}(0)\Psi(\infty)\rangle_{\mathbb{H}}.$$

The primary field $\Psi(\infty)$ should be chosen such as to satisfy the physical condition $\Delta(0) = 0$. This formula establishes the relation between the multifractal spectrum of critical curves and the correlation functions. Explicitly, the correlation function reads $\langle e^{i\sqrt{2}\alpha_h\phi(r)} e^{i\sqrt{2}\alpha_{1,2}\phi(0)} e^{i\sqrt{2}(2\alpha_0-\alpha_h-\alpha_{1,2})\phi(\infty)}\rangle$, where $0, r, \infty$ are three points on the real axis. It remains to find how it scales with r.

To find the r-dependence of this correlation function, we use the fusion of vertex operators (18) and obtain the result (52) written in a compact and suggestive form:

$$\Delta(h) = 2\alpha_{1,2}\alpha_h, \quad \alpha_h = \alpha_0 + \sqrt{\alpha_0^2 + h}. \tag{59}$$

An immediate generalization to the statistics of the harmonic measure of n curves reaching the system boundary at the same point is obtained by replacing $\psi_{1,2} \to \psi_{1,n+1}$. Since $\alpha_{1,n+1} = -n\alpha_-/2 = n\alpha_{1,2}$, this immediately leads to

$$\Delta^{(n)}(h) = 2\alpha_{1,n+1}\alpha_h = n\Delta(h),$$

which is the same as eq. (53).

We also remark that nothing compels us to measure the electric field on the real axis. Instead of the boundary field $\mathcal{O}_h(r)$ we could take a bulk primary field $\mathcal{O}_{h'}(z,\bar{z})$ where $|z| = r$. The weight h' should be chosen such that when the holomorphic part $\mathcal{O}_{h'}(z)$ is fused with its image $\mathcal{O}_{h'}(\bar{z})$ the boundary field $\mathcal{O}_h(\frac{z+\bar{z}}{2})$ with weight h is recovered.

[4]We drop the absolute value since on the real axis w' is positive.

6.3.2 Boundary multifractal exponent from SLE

For comparison, we compute the multifractal spectrum of a single curve in the SLE approach (see e.g. [14]).

In SLE we must compute $\prec w_t'(r)^h \succ$ in the limit $t \to \infty$ since the size of the curve $L \sim \sqrt{t}$. With the choice of the primary field as in (42) the eq. (41) at $t \to \infty$ reads

$$\Big(\frac{\kappa}{2}\partial_r^2 + \frac{2}{r}\partial_r - \frac{2h}{r^2}\Big) \prec w_t'^h(r)\succ = 0.$$

The solution is a power law $\prec w_t'(r)^h \succ \sim r^{2\alpha_{1,2}\alpha_h}$, in agreement with (59).

6.3.3 Calculation of higher boundary multifractal exponents

We consider n non-intersecting critical curves growing from the origin on the boundary (the real axis). It will be convenient to assume that the curves end somewhere in the bulk thus forming a boundary star (e.g. the third picture in Fig. 11). Let $w(z)$ be the conformal map of the slit domain formed by the star to the upper half plane normalized at infinity.

We want to find the scaling of the average

$$\prec |w'(z_1)|^{h_1} \ldots |w'(z_{n+1})|^{h_{n+1}} \succ ,$$

where z_i's are all close to the origin, no two of them lying in the same sector. The latter condition will be automatically satisfied in the subsequent calculation due to the following: if in a particular realization some two points z_i and z_j happen to be in the same sector, then $w(z_i) - w(z_j) \to 0$ as the size of the star $L \to \infty$.

Since n curves starting from the origin on the boundary are produced by the operator $\psi_{1,n+1}(0)$ we now consider a boundary CFT correlation function with several "probes" of the harmonic measure:

$$C = \Big\langle \prod_{i=1}^{n+1} \mathcal{O}_{h_i'}(z_i, \bar{z}_i)\psi_{1,n+1}(0)\Psi(\infty)\Big\rangle_{\mathbb{H}}.$$

This correlation function is found as

$$C \sim \prod_i |z_i|^{4\alpha_{1,n+1}\alpha_i'} \prod_{i<j} |z_i - z_j|^{4\alpha_i'\alpha_j'} \prod_{i,j} (z_i - \bar{z}_j)^{2\alpha_i'\alpha_j'}. \tag{60}$$

When all z_i are at the same distance r from the origin it scales as

$$C \sim r^{4\alpha_{1,n+1}\sum_i \alpha_i' + 8\sum_{i<j}\alpha_i'\alpha_j' + 2\sum_i \alpha_i'^2}. \tag{61}$$

As before, this correlation function is equal to the statistical average of a certain correlation function in the fluctuating domain, and we further apply the uniformizing map $w(z)$ to transform this domain into the upper half plane $\mathbb{H}$:

$$C = \prec \prod_i |w'(z_i)|^{2h'_i} \Big\langle \prod_i \mathcal{O}_{h'_i}\big(w(z_i), \overline{w(z_i)}\big)\Psi(\infty)\Big\rangle_{\mathbb{H}} \succ \; . \tag{62}$$

Unlike eq. (58), the correlation function inside the average scales as:

$$\prod_i (w(z_i) - \overline{w(z_i)})^{2\alpha_i'^2} \prod_{i<j} |w(z_i) - w(z_j)|^{4\alpha'_i\alpha'_j} \prod_{i \neq j} \big(w(z_i) - \overline{w(z_j)}\big)^{2\alpha'_i\alpha'_j}, \tag{63}$$

where $\alpha'_i \equiv \alpha_{h'_i} = \alpha_0 + \sqrt{\alpha_0^2 + h'_i}$. We specifically separated the $i = j$ terms since only they contribute to the short-distance behavior. All the other terms insure that the realizations of the curves in which any two points z_i end up in the same sector are suppressed since the distances $w(z_i) - w(z_j)$ are then small. Then the short-distance dependence of eq. (62) is

$$C \sim \prec \prod_i |w'(z_i)|^{2h'_i} (w(z_i) - \overline{w(z_i)})^{2\alpha_i'^2} \succ \; .$$

Insofar as the scaling with r is concerned, we further approximate $w(z_i) - \overline{w(z_i)} \sim |z_i||w'(z_i)| \sim r|w'(z_i)|$. This gives

$$C \sim r^{2\sum_i \alpha_i'^2} \prec \prod_i |w'(z_i)|^{2h'_i + 2\alpha_i'^2} \succ \; . \tag{64}$$

We denote the exponents inside the average

$$h_i = 2h'_i + 2\alpha_i'^2. \tag{65}$$

These are the weights of operators whose conformal charges $\alpha_{h_i} = \alpha_0 + \sqrt{\alpha_0^2 + h_i}$ are just $2\alpha'_i$.

Finally, comparing eqs. (61) and (64), we get the result

$$\prec |w'(z_1)|^{h_1} \ldots |w'(z_{n+1})|^{h_{n+1}} \succ \propto r^{\Delta^{(n)}(h_1, \ldots h_{n+1})},$$

with the higher multifractal exponent

$$\Delta^{(n)}(h_1, \ldots h_{n+1}) = 2\alpha_{1,n+1} \sum_{i=1}^{n+1} \alpha_{h_i} + 2 \sum_{i<j}^{n+1} \alpha_{h_i}\alpha_{h_j},$$

which is the formula (55).

If z_i's are kept independent the computation becomes more complicated. It can be done to the end in the case of a single curve. In this case the points can be taken on the boundary and C reduces to a four-point function which contains $\psi_{1,2}$:

$$C(x_1, x_2) \sim \prec w'(x_1)^{h_1} w'(x_2)^{h_2} \succ \sim \langle \mathcal{O}_{h_1}(x_1) \mathcal{O}_{h_2}(x_2) \psi_{1,2}(0) \Psi(\infty) \rangle_{\mathbb{H}}, \quad (66)$$

where x_1 and x_2 are two points on the real axis. Since $\psi_{1,2}$ is degenerate on level 2, C satisfies the equation (28). Same result can be obtained within the SLE approach. Up to a normalization the solution is [3]

$$C \sim x_1^{\Delta(h_1)} x_2^{\Delta(h_2)} (x_1 - x_2)^{\Delta - \Delta(h_1) - \Delta(h_2)} F\Big(\frac{x_1}{x_1 - x_2}\Big), \quad (67)$$

$$\Delta = h_\infty - h_{1,2} - h_1 - h_2,$$

where h_∞ is the weight of $\Psi(\infty)$ and $F(z)$ satisfies a hypergeometric equation:

$$\frac{\kappa}{4} z(1-z) F''(z) + \Big[1 + \frac{\kappa}{2}\Delta(h_1) - \Big(2 + \frac{\kappa}{2}\Delta(h_1) + \frac{\kappa}{2}\Delta(h_2)\Big) z\Big] F'(z)$$
$$- (\Delta(h_1, h_2) - \Delta) F(z) = 0.$$

One of the two possible solutions of this equation diverges at small z and the other one is regular. If $x_2 \to \infty$ the dependence of the correlation function on x_1 should be $C(x_1, \infty) \sim \langle \mathcal{O}_{h_1} \psi_{1,2} \Psi(\infty) \rangle \sim x_1^{\Delta(h_1)}$. We therefore should pick the regular solution of the hypergeometric equation. Then for C to be symmetric with respect to $(x_1, h_1) \leftrightarrow (x_2, h_2)$, $F(z)$ should be a constant. A constant is a solution of the hypergeometric equation only if the weight at infinity is fixed by setting

$$\Delta = \Delta(h_1, h_2).$$

We then obtain the formula (57).

We note that the function $C(x_1, x_2)$ as defined in eq. (66) is real and positive for any x_1, x_2. Therefore, the constant F must have such a phase as to make the expression (67) positive.

6.3.4 Calculation of bulk multifractal exponents

Calculation of the bulk multifractal behavior is done in much the same way as on the boundary, so we go straight to the general case of higher bulk exponents. The method of SLE is not yet developed for this case.

Let the critical system occupy the whole complex plane and be conditioned to have n critical curves growing from a single point in which we place the origin $z=0$. We will assume that $z=0$ is the only common point of these curves, since the local results around this point are unaffected by the curves' behavior at large distances. The curves then form a star. We define the conformal map $w(z)$ of the slit domain formed by the star to the exterior of a unit circle (the second picture in Fig. 11) and normalize $w(z)$ at infinity as before: $w(\infty)=\infty$, $w'(\infty)=\rho^{-1}$, where ρ is the conformal radius of the star.

Close to the origin the curves divide the plane into n sectors. We consider a quantity

$$\prec |w'(z_1)|^{h_1} \ldots |w'(z_n)|^{h_n} \succ \,,$$

where z_i's are points close to the origin, no two of them lying in one sector. As before, this is not a serious constraint because if some two points z_i and z_j happen to be in the same sector, $w(z_i)-w(z_j)\to 0$ when the size of the star $L\to\infty$.

Since n curves starting from the origin in the bulk are produced by the operator $\psi_{0,n/2}(0)$, we consider the correlation function

$$C_{\text{bulk}} = \big\langle \prod_{i=1}^{n} \mathcal{O}_{h_i'}(z_i,\bar z_i)\psi_{0,n/2}(0)\Psi(\infty)\big\rangle,$$

where h_i' are related to h_i as in (65): h_i have twice larger holomorphic charges:

$$2\alpha_i' = \alpha_{h_i} = \alpha_0 + \sqrt{\alpha_0^2 + h_i}.$$

The correlation function is found to be

$$C_{\text{bulk}} \sim \prod_i |z_i|^{4\alpha_{0,n/2}\alpha_i'} \prod_{i<j} |z_i - z_j|^{4\alpha_i'\alpha_j'} \sim r^{4\alpha_{0,n/2}\sum_i \alpha_i' + 4\sum_{i<j}\alpha_i'\alpha_j'},$$

assuming that all z_i are of the order of r. Proceeding exactly as in the boundary case we use the two-step averagin to rewrite the correlation function C_{bulk} through the conformal map of the fluctuating star:

$$C_{\text{bulk}} = \prec \prod_i |w'(z_i)|^{2h_i'} \big\langle \prod_i \mathcal{O}_{h_i'}\big(w(z_i), \overline{w(z_i)}\big)\Psi(\infty)\big\rangle_{\mathbb{C}\setminus\text{star}} \succ \,.$$

The correlation function inside the average is found to be the same as (63) although $w(z)$ is defined differently. Combining the results we obtain

$$\prec|w'(z_1)|^{h_1}\dots|w'(z_n)|^{h_n}\succ\,\sim r^{\Delta^{(n)}_{\rm bulk}(h_1,\dots h_n)},$$

with the higher bulk exponent

$$\Delta^{(n)}_{\rm bulk}(h_1,\dots h_n)=\sum_{i=1}^{n}\Delta^{(n)}_{\rm bulk}(h_i)+\sum_{i<j}^{n}\alpha_{h_i}\alpha_{h_j},$$

where

$$\Delta^{(n)}_{\rm bulk}(h)=2\alpha_{0,n/2}\alpha_h-\frac{1}{2}\alpha_h^2=(2\alpha_{0,n/2}-\alpha_0)\alpha_h-\frac{h}{2}$$

is the scaling exponent of a single $\prec|w'(z)|^h\succ$ in the presence of n critical curves in the bulk. These are the results quoted in eqs. (54, 56).

7 Conclusions

In this work we attempted to clarify the subtle relation between the Gaussian Bose field description of critical behavior of statistical models and stochastic geometry of critical curves occurring in these systems. We further studied the relation between two complementary approaches: the algebraic approach of conformal field theory and a more direct approach of stochastic (Schramm-) Loewner evolution.

A natural extension of this work is a study of global properties of critical curves such as the distribution of the area and harmonic moments of a closed critical loop.

The critical curves considered in this paper all appear at 2D critical points described by CFTs with central charges $c \leqslant 1$. There are other CFTs which in addition to conformal symmetry have other symmetries, e.g. current algebras. These theories allow central charges $c > 1$.

We have recently extended the SLE approach to the Wess-Zumino model with the current algebra $\mathrm{SU}(2)_k$ [49]. In this case the SLE trace carries an additional spin degree of freedom. The analog of the Gaussian field takes values in the Lie group. The critical curves which appear in such a system can also be studied by the methods described here.

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References

[1] A. M. Polyakov, JETP Lett. **12**, 381 (1970).

[2] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984).

[3] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal field theory* (Springer, 1999).

[4] B. Duplantier, Phys. Rev. Lett. **84**, 1363 (2000); arXiv: cond-mat/9908314.

[5] B. Duplantier, in *Fractal geometry and applications*, Proc. Sympos. Pure Math. **72**, part 2, Amer. Math. Soc., 2004; arXiv: math-ph/0303034.

[6] E. Bettelheim, I. Rushkin, I. A. Gruzberg, and P. Wiegmann, Phys. Rev. Lett. **95**, 170602 (2005); arXiv: hep-th/0507115.

[7] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, Mod. Phys. Lett. A **3**, 819 (1988).

[8] O. Schramm, Israel J. Math. **118**, 221 (2000); arXiv: math.PR/9904022.

[9] M. Bauer and D. Bernard, in *String theory: from gauge interactions to cosmology*, 41, NATO Sci. Ser. II Math. Phys. Chem., **208**, Springer, Dordrecht, (2006); arXiv: cond-mat/0412372.

[10] M. Bauer and D. Bernard, Phys. Rep. **432**, 115 (2006); arXiv: math-ph/0602049.

[11] J. Cardy, Ann. Phys. **318**, 81 (2005); arXiv: cond-mat/0503313.

[12] I. A. Gruzberg, J. Phys. A **89**, 12601 (2006); arXiv: math-ph/0607046.

[13] W. Kager and B. Nienhuis, J. Stat. Phys. **115**, 1149 (2004); arXiv: math-ph/0312056.

[14] G. F. Lawler, *Conformally invariant processes in the plane*. Mathematical Surveys and Monographs, 114. American Mathematical Society, Providence, RI, 2005.

[15] G. F. Lawler, in *Random walks and geometry*, 261, Walter de Gruyter GmbH & Co. KG, Berlin 2004; available online at URL `http://www.math.cornell.edu/~lawler/papers.html`

[16] O. Schramm, in *Current developments in mathematics*, 2000, 233, Int. Press, Somerville, MA, 2001.

[17] O. Schramm, to appear in the ICM 2006 Madrid Proceedings, arXiv: math.PR/0602151.

[18] W. Werner, in *Lectures on probability theory and statistics*. Lecture Notes in Mathematics, **1840**. Springer-Verlag, Berlin, 2004; arXiv: math.PR/0303354.

[19] W. Werner, Probab. Surv. **2**, 145 (2005) (electronic); arXiv: math.PR/0307353.

[20] B. Nienhuis, in *Phase transitions and critical phenomena*, edited by C. Domb (Academic Press, 1987), vol. 11.

[21] J. L. Cardy, Nucl. Phys. **B240**, 514 (1984); **B324**, 581 (1989).

[22] M. Bauer, D. Bernard and J. Houdayer, J. Stat. Mech. (2005) P03001; arXiv: math-ph/0411038.

[23] J. Schulze, Nucl. Phys. **B489**, 580 (1997); arXiv: hep-th/9602177.

[24] S. Kawai, J. Phys. A **36**, 6547 (2003); arXiv: hep-th/0210032.

[25] H. Saleur, Phys. Rep. **184**, 177 (1989).

[26] B. Duplantier, Phys. Rep. **184**, 229 (1989).

[27] C. Vanderzande, *Lattice models of polymers*, Cambridge University Press, 1998.

[28] J. Kondev, Int. J. Mod. Phys. **B11**, 153 (1997); arXiv: cond-mat/9607181.

[29] J. Kondev, Phys. Rev. Lett. **78**, 4320 (1997); arXiv: cond-mat/9703113.

[30] O. Schramm and S. Sheffield, arxiv: math.PR/0605337.

[31] S. Rohde and O. Schramm, Ann. of Math. (2) **161**, 883 (2005); arXiv: math.PR/0106036.

[32] Vl. S. Dotsenko, V. A. Fateev, Nucl. Phys. **B240**, 312 (1984).

[33] M. Bauer and D. Bernard, in Proceedings of the NATO conference *"Conformal invariance and random spatial processes"*, Edinburgh (2003); arXiv: math-ph/0401019.

[34] M. Bauer and D. Bernard, Phys. Lett. B **543**, 135 (2002) ; arXiv: math-ph/0206028.

[35] M. Bauer and D. Bernard, Commun. Math. Phys. **239**, 493 (2003); arXiv: hep-th/0210015.

[36] M. Bauer and D. Bernard, Phys. Lett. B **583**, 324 (2004); arXiv: math-ph/0310032.

[37] L. P. Kadanoff, J. Phys. A **11**, 1399 (1978).

[38] J. Dai, R.G. Leigh, and J. Polchinski, Mod. Phys. Lett. A **4**, 2073 (1989).

[39] O. Schramm and S. Sheffield, unpublished; see Sheffield's lectures at `http://www.fields.utoronto.ca/audio/05-06/#percolation_SLE`

[40] L. V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill, New York (1973).

[41] B. Øksendal, *Stochastic differential equations*, Springer-Verlag, Berlin, 2003.

[42] F. C. Klebaner, *Introduction to stochastic calculus with applications*, Imperial College Press, London, 1998.

[43] R. Friedrich and W. Werner, Comm. Math. Phys. **243**, 105 (2003); arXiv: math-ph/0301018.

[44] B. Doyon, V. Riva, and J. Cardy, Comm. Math. Phys. **268**, 687 (2006); arXiv: math-ph/0511054.

[45] J. B. Garnett and D. E. Marshall, *Harmonic measure*, Cambridge University Press, 2005.

[46] C. Pommerenke, *Boundary behaviour of conformal maps*, Springer, 1992.

[47] N. G. Makarov, Proc. London Math. Soc. **51**, 369 (1985).

[48] N. G. Makarov, St. Petersburg Math. J. **10**, 217 (1999).

[49] E. Bettelheim, I. A. Gruzberg, A. W. W. Ludwig, and P. Wiegmann, Phys. Rev. Lett. **95**, 251601 (2005); arXiv: hep-th/0503013.